

Energy norm error estimates for averaged discontinuous Galerkin methods: multidimensional case

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Abstract

A mathematical analysis is presented for a class of interior penalty (IP) discontinuous Galerkin approximations of elliptic boundary value problems. In the framework of the present theory one can derive some overpenalized IP bilinear forms in a natural way avoiding any heuristic choice of fluxes and penalty terms. The main idea is to start from bilinear forms for the local average of discontinuous approximations which are rewritten using the theory of distributions. It is pointed out that a class of overpenalized IP bilinear forms can be obtained using a lower order perturbation of these. Also, error estimations can be derived between the local averages of the discontinuous approximations and the analytic solution in the H^1 -seminorm. Using the local averages, the analysis is performed in a conforming framework without any assumption on extra smoothness for the solution of the original boundary value problem.

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1 Introduction

Discontinuous Galerkin (dG) methods have been introduced and used from the end of the seventies, first for linear transport problems. Later this was generalized to elliptic boundary value problems and nowadays it is available for the numerical solution of almost all kind of problems based on PDE's.

These methods have proved their usefulness in several simulations of real-life phenomena [8], [15], [27]. The most favorable property of the corresponding numerical methods is that the local mesh refinement can easily be performed giving rise to efficient adaptive strategies.

An important milestone in the systematic analysis of dG methods for the elliptic boundary value problems was the paper [1]. This pioneering work served as a basis of the consecutive works concerning a priori and a posteriori error estimates [20], *hp*-adaptive methods [18], time dependent problems [6]. For an up-to-date summary of the theoretical achievements for dG methods we refer the recent monograph [10] and for implementation issues the monographs [16] and [25].

At the same time, the above analysis should be improved in some aspects. First, which can be considered as a didactic issue, the choice of the corresponding bilinear forms would deserve more motivation. After recasting the elliptic problem in a mixed form, numerical fluxes and penalty terms are defined which lead to different bilinear forms. No a priori suggestion or motivation (on a physical

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basis) is mentioned to propose an appropriate choice of the fluxes. A similar situation arises when penalty terms are defined.

The second issue is the assumption on extra-regularity of the analytic solution. This problem was solved in the meantime: in [13] the author developed an analysis based on a Strang type lemma [11], which could successfully deal with the non-conformity of the dG type approximation.

The most important issue is the norm for the convergence. The choice of the bilinear form implies a mesh-dependent norm, which is a real mathematical artifact. The convergence is proved with respect to this norm or in a weaker, *e.g.*, in the L_2 -norm. At the same time, in the corresponding real-life problems the natural norm is usually the H^1 -norm (or seminorm). Note that there are some achievements which point out the usefulness of the interior penalty (IP) methods. For these methods, one can obtain convergence in the so-called BV norm which does not depend on the actual mesh [4], [9] and can be related to broken Sobolev norms.

The aim of the present work is to contribute to the mathematics of the dG methods for elliptic boundary value problems by proposing an alternative of the commonly used theoretical basis in [1]. In particular, we derive overpenalized interior penalty bilinear forms in a natural way avoiding the notion of numerical fluxes or recasting them into a mixed form. The new idea is to use the local average of the discontinuous approximation from the beginning. The main benefit of the analysis is that it can be done in an H^1 -conforming framework such that one can prove the quasi optimal convergence of the local average with respect to the natural H^1 -seminorm for Dirichlet problems. This work is a generalization of the paper [7] concerning the one-dimensional case.

The idea to use *postprocessing* (or smoothing or filtering) for dG approximations has already appeared in the literature [5]. In the last years, many related results have been achieved: involved algorithms were developed for linear hyperbolic problems in [21] and their accuracy-increasing property was verified also for advection-diffusion problems with respect to negative Sobolev norms [19]. The accurate computation of the corresponding convolutions is challenging, see the recent developments in [22] and [23].

The setup of the article is as follows. After some preliminaries we give the bilinear form for the averaged approximation, which still contains convolution terms. We then expand the terms and point out that with a lower-order perturbation an overpenalized IP bilinear form can be obtained. This result is given in Theorem 1. Based on this, we can state the closedness of the approximation from the new bilinear form and the one arising from the overpenalized IP bilinear form, see Theorem 2. Finally, in Theorem 3, an optimal convergence rate for the averaged overpenalized IP approximation is proved in the H^1 (semi)norm. The only tool we use beyond the standard armory of the finite element analysis is a bit of distribution theory.

2 Mathematical preliminaries

We investigate the finite element solution of the elliptic boundary value problem

$$\begin{cases} -\Delta u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Omega \subset \mathbb{R}^d \\ u(\mathbf{x}) = 0 & \mathbf{x} \in \partial\Omega, \end{cases} \quad (1)$$

where Ω is a polyhedral Lipschitz-domain and $g \in L_2(\Omega)$ is given.

The finite element approximation is computed on a non-degenerated simplicial mesh \mathcal{T}_h with the mesh parameter h . The symbol \mathcal{F} denotes the set of interelement faces. For the numerical solution we use the finite element space

$$\mathbb{P}_{h,\mathbf{k}} = \{u \in L_2(\Omega) : u|_K \in P_{k_j}(\Omega_j) \text{ for all } \Omega_j \in \mathcal{T}_h\},$$

where $\mathbf{k} = (k_1, k_2, \dots)$ and $\mathbb{P}_{k_j}(\Omega_j)$ denotes the linear space of polynomials of total degree k_j on the subdomain Ω_j . This notation will also be used for interelement faces and for balls instead of Ω_j . We also make use of the conventional notation $\{\!\{ \cdot \}\!\} : \mathbb{P}_{h,\mathbf{k}} \rightarrow L_2(\mathcal{F})$ and $\llbracket \cdot \rrbracket : \mathbb{P}_{h,\mathbf{k}} \rightarrow L_2(\mathcal{F})$ for the average and jump operators which are given on each interelement face $f_\Omega = \bar{\Omega}_+ \cap \bar{\Omega}_-$ with

$$\{\!\{ v \}\!\}_{f_\Omega}(\mathbf{x}) = \frac{1}{2}(v(\mathbf{x}_+) + v(\mathbf{x}_-)) \quad \text{and} \quad \llbracket v \rrbracket_{f_\Omega}(\mathbf{x}) = \boldsymbol{\nu}_+ v(\mathbf{x}_+) + \boldsymbol{\nu}_- v(\mathbf{x}_-).$$

Here $\boldsymbol{\nu}_\pm$ denotes the outward normal of Ω_\pm and $v(\mathbf{x}_\pm) = \lim_{\Omega_\pm \ni \mathbf{x}_n \rightarrow \mathbf{x}} v(\mathbf{x}_n)$. On each boundary face $f \subset \partial\Omega$ we simply define

$$\{\!\{ v \}\!\}_f(\mathbf{x}) = v(\mathbf{x}) \quad \text{and} \quad \llbracket v \rrbracket_f(\mathbf{x}) = \boldsymbol{\nu}(\mathbf{x})v(\mathbf{x}).$$

The $L_2(\Omega^*)$ norm on a generic domain Ω^* will be denoted with $\|\cdot\|_{\Omega^*}$ and the corresponding scalar product with $(\cdot, \cdot)_{\Omega^*}$. In case of $\Omega^* = \Omega$ or if the support of the terms is given, we omit the subscript. Similar notation is applied for the scalar product and the corresponding L_2 norm on \mathcal{F} and on a single interelement face f .

With these, the most popular dG approximation of u in (1) is the so-called symmetric interior penalty dG method which is given with the bilinear form $a_{\text{IP}} : \mathbb{P}_{h,\mathbf{k}} \times \mathbb{P}_{h,\mathbf{k}} \rightarrow \mathbb{R}$ as follows:

$$a_{\text{IP}}(u, v) = (\nabla_h u, \nabla_h v) - \sum_{f \in \mathcal{F}} (\{\!\{ \nabla_h u \}\!\}, \llbracket v \rrbracket)_f + (\{\!\{ \nabla_h v \}\!\}, \llbracket u \rrbracket)_f + \sum_{f \in \mathcal{F}} \sigma_h(\llbracket u \rrbracket, \llbracket v \rrbracket)_f, \quad (2)$$

where ∇_h denotes the piecewise gradient on the subdomains in \mathcal{T}_h and $\sigma_h \in \mathbb{R}$ denotes a penalty parameter, which is proportional with $(\text{diam } f)^{-1}$ in the conventional setting. We will also use the notation $\nabla_f \llbracket u \rrbracket$ for the gradient of the jump functions defined on the interelement face f .

The notation $\lambda_d(\cdot)$ will be used to the d -dimensional Lebesgue measure. For the local average we use the piecewise constant function $\eta_h : \mathbb{R}^n \rightarrow \mathbb{R}$ depending also on the parameter $s > 1$ with

$$\eta_h(\mathbf{x}) = \begin{cases} \frac{1}{B_{h^s,d}} & |\mathbf{x}| \leq h^s \\ 0 & |\mathbf{x}| > h^s, \end{cases}$$

where $B(\mathbf{x}, r)$ denotes the closed ball with radius r centered at \mathbf{x} and $B_{h^s,d} = \lambda_d(B(\mathbf{0}, h^s))$. The analysis makes use only two properties of η_h : this is symmetric with respect to the origin and $\int_{\mathbb{R}^d} \eta_h = 1$ such that $\eta_h * u$ is the local average of the function $u : \mathbb{R}^d \rightarrow \mathbb{R}$. Also, a straightforward computation gives that $\text{supp } \eta_h * \eta_h = B(\mathbf{0}, 2h^s)$ and $\int_{B(\mathbf{0}, 2h^s)} \eta_h * \eta_h = 1$. These facts will be used without further reference.

The analysis of the conforming approach will be carried out in the space

$$\mathbb{P}_{h,\mathbf{k},s} = \{\eta_h * u_0|_{\Omega_h} : u_0 \text{ is the zero extension of } u \in \mathbb{P}_{h,\mathbf{k}}\},$$

where $\Omega_h = \{\mathbf{x} \in \mathbb{R}^d : d(\mathbf{x}, \Omega) < h^s\}$. Obviously, $\mathbb{P}_{h,\mathbf{k},s} \subset H_0^1(\Omega_h)$. We use the notation $\Omega_{j,h}$ in a similar sense and $\tilde{\Omega}_j = \text{int } \{\bar{\Omega}_k \in \mathcal{T}_h : \bar{\Omega}_j \cap \bar{\Omega}_k \neq \emptyset\}$ for the patch of Ω_j .

To extend the standard scaling arguments we first define a reference set \mathcal{K} of neighboring simplex pairs (K_+, K_-) having the interelement face $f = \bar{K}_+ \cap \bar{K}_-$ such that the following conditions hold:

- $f \subset 0 \times \mathbb{R}^{d-1}$ and one vertex of f is $\mathbf{0} \in \mathbb{R}^d$
- the maximum edge-length of f is *one*
- K_+ and K_- satisfy the condition on non-degeneracy.

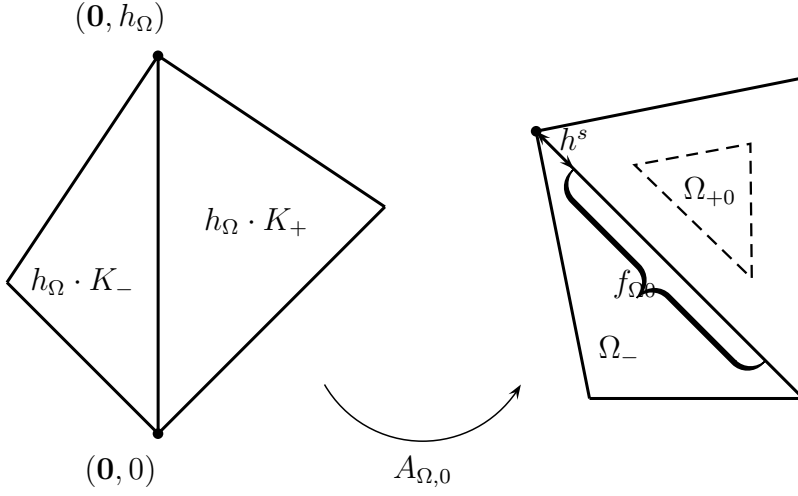


Figure 1: The transformation of the reference subdomain pair, the interior domain Ω_{+0} and the interior face $f_{\Omega 0}$ in the 2-dimensional case.

Then for any neighboring subdomains $\Omega_+, \Omega_- \in \mathcal{T}_h$ there is a pair $(K_+, K_-) \in \mathcal{K}$ and an affine linear map $A_\Omega : K_+ \cup K_- \rightarrow \Omega_+ \cup \Omega_-$ with $A_\Omega(K_+) = \Omega_+$ and $A_\Omega(K_-) = \Omega_-$, moreover

$$A_\Omega(\mathbf{x}) = A_{\Omega,0}(h_\Omega \mathbf{x}), \quad (3)$$

where h_Ω denotes the maximum edge length of f_Ω and $A_{\Omega,0}$ is an isometry; see also Fig. 1.

Accordingly, for any $v \in \mathbb{P}_{h,\mathbf{k}}(\Omega_+ \cup \Omega_-)$ the function $v_0 := v \circ A_\Omega \in \mathbb{P}_{h,\mathbf{k}}(K_+ \cup K_-)$, moreover, using (3) the following equalities are valid:

$$\llbracket v_0 \rrbracket(\mathbf{x}) = \llbracket v \rrbracket(A_\Omega(\mathbf{x})) \quad \text{and} \quad \eta_{h_0} * v_0(\mathbf{x}) = \eta_{h_0 h^{\frac{1}{s}}} * v(A_\Omega \mathbf{x}), \quad (4)$$

whenever the operation $\eta_{h_0} *$ makes sense.

We also use the notation $h_\Omega \cdot K_\pm = \{h_\Omega \mathbf{x} : \mathbf{x} \in K_\pm\}$ and similarly $h_\Omega \cdot f$ and introduce the interior domain $\Omega_{j0} = \{\mathbf{x} \in \Omega_j : B(\mathbf{x}, h^s) \subset \Omega_j\}$ and the interior face $f_0 \subset f$ similarly.

The space $\text{BV}(\Omega)$ of real valued functions on Ω with bounded variations is defined with

$$\text{BV}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : \sup_{\substack{\phi \in [C_c^1(\Omega)]^d \\ \|\phi\|_\infty = 1}} \int_\Omega u \nabla \cdot \phi := |u|_{\text{BV}} < \infty \right\}$$

and is equipped with the seminorm $|\cdot|_{\text{BV}}$, where $\|\cdot\|_\infty$ denotes the maximum norm on $C_c^1(\Omega)$. This seminorm can also be given as

$$|\cdot|_{\text{BV}} = \int_\Omega d|\partial u|,$$

where $|\partial u|$ is the Radon measure generated by the distributional derivative of u .

The dual pairing between a distribution S and a test function ϕ denoted using angle brackets: $\langle S, \phi \rangle$.

In the estimates, the notation $g_1 \lesssim g_2$ means the existence of a constant C - which does not depend on the mesh parameter but possibly on the local polynomial degree - such that $g_1 \leq C \cdot g_2$. We also use the notation $g_1 \sim g_2$ provided that both $g_1 \lesssim g_2$ and $g_2 \lesssim g_1$ are satisfied.

3 Results

The basic idea of the present analysis is to find a smoothed dG approximation immediately. In this case, in the background we can compute with discontinuous basis functions in $\mathbb{P}_{h,\mathbf{k}}$ and still have the freedom to choose them independently on the neighboring subdomains. On the other hand, as we compute conforming approximations, we can use the entire armory of the classical finite element analysis.

The smoothed (or averaged) dG approximation consists of finding $\eta_h * u_h \in \mathbb{P}_{h,\mathbf{k},s}$ such that for all $\eta_h * v_h \in \mathbb{P}_{h,\mathbf{k},s}$ we have

$$a_\eta(u_h, v_h) := a_\eta^+(\eta_h * u_h, \eta_h * v_h) := (\nabla(\eta_h * u_h), \nabla(\eta_h * v_h)) = (g_0, \eta_h * v_h), \quad (5)$$

where the bilinear forms $a_\eta : \mathbb{P}_{h,\mathbf{k}} \times \mathbb{P}_{h,\mathbf{k}} \rightarrow \mathbb{R}$ and $a_\eta^+ : \mathbb{P}_{h,\mathbf{k},s} \times \mathbb{P}_{h,\mathbf{k},s} \rightarrow \mathbb{R}$ are defined by (5) and g_0 denotes the zero extension of g to Ω_h . Whenever the spaces $\mathbb{P}_{h,\mathbf{k},s} \not\subset H_0^1(\Omega)$ we call the method H^1 -conforming since each space is in $H_0^1(\Omega_h)$.

We make use of the following inequalities, which can be proved using simple scaling arguments.

Proposition 1 *We have the following inequalities:*

$$\max_{B(\mathbf{0}, h^s)} |u| \sim h^{-\frac{sd}{2}} \|u\|_{B(\mathbf{0}, h^s)} \quad \forall u \in \mathbb{P}_k(B(\mathbf{0}, h^s)), \quad (6)$$

$$\max_f \llbracket u \rrbracket \lesssim h^{1-d} \int_f |\llbracket u \rrbracket| \quad \forall \llbracket u \rrbracket \in \mathbb{P}_k(f), \quad (7)$$

$$\max_K |\nabla^2 u| \lesssim h^{-\frac{d}{2}-2} \|u\|_K \quad \forall u \in \mathbb{P}_k(K), \quad (8)$$

$$\max_f \nabla_f^2 \llbracket u \rrbracket \lesssim h^{-d-1} \int_f |\llbracket u \rrbracket| \quad \forall \llbracket u \rrbracket \in \mathbb{P}_k(f), \quad (9)$$

$$\|\nabla u\|_{B(\mathbf{0}, h^s)} \lesssim h^{\frac{(s-1)d}{2}} \|\nabla u\|_{B(\mathbf{0}, h)} \lesssim h^{-1} h^{\frac{(s-1)d}{2}} \|u\|_{B(\mathbf{0}, h)} \quad \forall u \in \mathbb{P}_k(B(\mathbf{0}, h)). \quad \square \quad (10)$$

We need also an estimate between the discontinuous function $\nabla_h u$ and its local average $\eta_h * \nabla_h u$ with a convergence rate depending on h . For this a Taylor expansion is developed about all $\mathbf{x} \in \Omega_{j0}$ giving for an arbitrary $\mathbf{y} \in \Omega_j$ that

$$u(\mathbf{y}) = u(\mathbf{x}) + \nabla u(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) + \frac{1}{2} \nabla^2 u(\boldsymbol{\xi}_{\mathbf{y}}) (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \quad (11)$$

for some $\boldsymbol{\xi}_{\mathbf{y}}$ in the section (\mathbf{x}, \mathbf{y}) . Integrating both sides over $B(\mathbf{x}, h^s)$ yields

$$B_{h^s,d} \cdot (\eta_h * u(\mathbf{x})) = B_{h^s,d} \cdot u(\mathbf{x}) + \int_{B(\mathbf{x}, h^s)} \frac{1}{2} \nabla^2 u(\boldsymbol{\xi}_{\mathbf{y}}) (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})$$

and therefore

$$\eta_h * u(\mathbf{x}) - u(\mathbf{x}) = \frac{1}{2 \cdot B_{h^s,d}} \int_{B(\mathbf{0}, h^s)} \nabla^2 u(\boldsymbol{\xi}_{\mathbf{y}}) |\mathbf{y}|^2 d\mathbf{y}. \quad (12)$$

Proposition 2 *For all $u \in \mathbb{P}_{h,\mathbf{k}}$ and subdomain Ω_j we have*

$$\|\nabla_h u - \eta_h * \nabla_h u\|_{\Omega_j} \lesssim h^{\frac{s-1}{2}} \|\nabla_h u\|_{\tilde{\Omega}_j}.$$

Proof: We first use the triangle inequality

$$\begin{aligned} \|\nabla_h u - \eta_h * \nabla_h u\|_{\Omega_j} &\leq \|\nabla_h u - \eta_h * \nabla_h u\|_{\Omega_{j0}} + \|\nabla_h u - \eta_h * \nabla_h u\|_{\Omega_j \setminus \Omega_{j0}} \\ &\leq \|\nabla_h u - \eta_h * \nabla_h u\|_{\Omega_{j0}} + \|\nabla_h u\|_{\Omega_j \setminus \Omega_{j0}} + \|\eta_h * \nabla_h u\|_{\Omega_j \setminus \Omega_{j0}}, \end{aligned} \quad (13)$$

where the contributions are estimated separately. We obviously have the estimate $\lambda_d(\Omega_j \setminus \Omega_{j0}) \lesssim h^s h_{\Omega_j}^{d-1}$ such that a simple scaling argument gives

$$\|\nabla_h u\|_{\Omega_j \setminus \Omega_{j0}} \lesssim h^{\frac{s-1}{2}} \|\nabla_h u\|_{\Omega_j}. \quad (14)$$

This also implies, using (10) in the second line with $s = 1$ that

$$\begin{aligned} \|\eta_h * \nabla_h u\|_{\Omega_j \setminus \Omega_{j0}}^2 &\leq \lambda_d(\Omega_j \setminus \Omega_{j0}) \max_{\tilde{\Omega}_j} |\nabla_h u|^2 \\ &\leq h_{\Omega_j}^{d-1} h^s \max_{\tilde{\Omega}_j} |\nabla_h u|^2 \leq h_{\Omega_j}^{d-1} h^s h_{\Omega_j}^{-d} \|\nabla_h u\|_{\tilde{\Omega}_j}^2 = h^{s-1} \|\nabla_h u\|_{\tilde{\Omega}_j}^2. \end{aligned} \quad (15)$$

Finally, combining the inequalities in (12) and (8) we arrive at the estimate

$$\begin{aligned} |\nabla_h u - \eta_h * \nabla_h u|_{\Omega_{j0}} &\leq \frac{1}{2 \cdot \lambda_d(B(\mathbf{x}, h^s))} \max_{\mathbf{y} \in \Omega_j} |\nabla^3 u(\mathbf{y})| \int_{B(\mathbf{0}, h^s)} |\mathbf{y}|^2 d\mathbf{y} \\ &\lesssim h^{-sd} \max_{\mathbf{y} \in \Omega_j} |\nabla^3 u(\mathbf{y})| h^{s(d+2)} \lesssim h^{2s} h^{-\frac{d}{2}-2} \|\nabla u\|_{\Omega_j}. \end{aligned}$$

Therefore, using (6) we obtain

$$\|\nabla_h u - \eta_h * \nabla_h u\|_{\Omega_{j0}} \leq h^{2s-2} \|\nabla_h u\|_{\Omega_j}. \quad (16)$$

The estimates (14), (15) and (16) with (13) imply then the inequality in the proposition. \square

Remark: For functions $v \in C^2(\mathbb{R}^d)$ one can easily estimate the difference in Proposition 2. Moreover, it turns out that the convergence rate of the difference $\int_{\mathbb{R}^d} |\eta_h * v|^2 - |v|^2$ characterizes the Sobolev space $H^1(\mathbb{R}^n)$, see [24].

The chief problem in the estimations with convolution terms is that the scaling arguments can not be applied in a straightforward way. Whenever we use polynomial spaces the function space $\{\eta_h * v : v \in \mathbb{P}_{h,\mathbf{k}}, 0 < h < h_0\}$ is infinite dimensional, which makes the following proofs non-trivial.

Proposition 3 *There exists $h_0 > 0$ such that for all h with $h^{1-\frac{1}{s}} < h_0$ and $v \in \mathbb{P}_{h,\mathbf{k}}(\Omega_+ \cup \Omega_-)$ we have*

$$\int_{f\Omega} |\llbracket v \rrbracket| \lesssim \int_{\Omega_+ \cup \Omega_-} |\nabla(\eta_h * v)| \quad (17)$$

and for $s \geq \frac{3}{2}$

$$\int_{f\Omega} |\llbracket v \rrbracket| \lesssim h^{\frac{d}{2}} \sqrt{\int_{\Omega_+ \cup \Omega_-} |\nabla(\eta_h * v)|^2}. \quad (18)$$

The corresponding proof is postponed to the Appendix.

3.1 The bilinear form

To give the bilinear form (5) in a more explicit form, we first need some identities for distributional derivatives.

We first decompose the gradient of a function $u \in \mathbb{P}_{h,\mathbf{k}}(\Omega)$ as follows.

Lemma 1 *For all $u \in \mathbb{P}_{h,\mathbf{k}}(\Omega)$ we have*

$$\nabla u = \nabla_h u + \llbracket u \rrbracket_{\mathcal{D}}$$

in the sense of distributions, i.e. $\llbracket u \rrbracket_{\mathcal{D}} \in [\mathcal{D}^3(\Omega)]^$ is a distribution with*

$$\langle \llbracket u \rrbracket_{\mathcal{D}}, \phi \rangle = - \sum_{f \in \mathcal{F}} \int_f \llbracket u \rrbracket_f \cdot \phi := - \int_{\mathcal{F}} \llbracket u \rrbracket \cdot \phi = -(\llbracket u \rrbracket, \phi)_{\mathcal{F}}.$$

Proof: Obviously, for all $\phi \in [\mathcal{D}(\Omega)]^3$ we have

$$\begin{aligned} \langle \nabla u, \phi \rangle &= -\langle u, \nabla \cdot \phi \rangle = - \sum_{\Omega_j \in \mathcal{T}_h} \int_{\Omega_j} u \nabla \cdot \phi = \sum_{\Omega_j \in \mathcal{T}_h} \int_{\Omega_j} \nabla u \cdot \phi - \sum_{\Omega_j \in \mathcal{T}_h} \int_{\partial \Omega_j} u|_{\Omega_j} \nu_j \cdot \phi \\ &= \sum_{\Omega_j \in \mathcal{T}_h} \int_{\Omega_j} \nabla u \cdot \phi - \sum_{f \in \mathcal{F}} \int_f \llbracket u \rrbracket_f \cdot \phi|_f, \end{aligned}$$

which proves the statement. \square

Remarks: The decomposition in Lemma 1 is indeed a Lebesgue decomposition [14] of the Radon measure corresponding to the distributional derivative ∇u , which can be considered as a special case of the one in [26]. The role of the jump terms in this context is analyzed in [2], Section 10.

The symbol $\llbracket \cdot \rrbracket_{\mathcal{D}}$ can be understood both as a distribution supported on the interelement faces and the singular measure in the corresponding Lebesgue decomposition. The connection between $\llbracket u \rrbracket_{\mathcal{D}}$ with classical function $\llbracket u \rrbracket$ is highlighted in Lemma 1.

The negative sign is a weakness of the conventional notation. This is already transparent in the one-dimensional case: whenever the Heaviside step function $H : \mathbb{R} \rightarrow \mathbb{R}$ is increasing, by definition we have $\llbracket H \rrbracket(0) = -1$.

For the consecutive derivations we need also an identity regarding the convolution of distributions.

Lemma 2 *For all $u \in \mathbb{P}_{h,\mathbf{k}}$ the convolution $\eta_h * \llbracket u \rrbracket_{\mathcal{D}}$ is regular, which will be identified with the corresponding locally integrable function. With this, for all bounded function $\mathbf{w} : \Omega \rightarrow \mathbb{R}^3$ we have*

$$\langle \eta_h * \llbracket u \rrbracket_{\mathcal{D}}, \mathbf{w} \rangle = (\llbracket u \rrbracket, \eta_h * \mathbf{w})_{\mathcal{F}}.$$

Proof: Since both η_h and $\llbracket u \rrbracket$ are compactly supported, we get by definition (see [17], Definition 2.1) and by Lemma 1 that for each $\phi \in [C_0^\infty(\Omega)]^3$ the following equality is valid:

$$\begin{aligned} \langle \eta_h * \llbracket u \rrbracket_{\mathcal{D}}, \phi \rangle &= \langle \llbracket u \rrbracket_{\mathcal{D}}, \mathbf{y} \rightarrow \eta_h(\mathbf{x} \rightarrow \phi(\mathbf{x} + \mathbf{y})) \rangle = \langle \llbracket u \rrbracket_{\mathcal{D}}, \mathbf{y} \rightarrow \int_{\mathbb{R}^d} \eta_h(\mathbf{x}) \phi(\mathbf{x} + \mathbf{y}) \, d\mathbf{x} \rangle \\ &= - \int_{\mathcal{F}} \llbracket u \rrbracket(\mathbf{y}) \int_{\mathbb{R}^d} \eta_h(\mathbf{x}) \phi(\mathbf{x} + \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} = - \int_{\mathcal{F}} \llbracket u \rrbracket(\mathbf{y}) \int_{\mathbb{R}^d} \eta_h(\mathbf{z} - \mathbf{y}) \phi(\mathbf{z}) \, d\mathbf{z} \, d\mathbf{y} \\ &= - \int_{\mathcal{F}} \llbracket u \rrbracket(\mathbf{y}) \int_{\mathbb{R}^d} \eta_h(\mathbf{y} - \mathbf{z}) \phi(\mathbf{z}) \, d\mathbf{z} \, d\mathbf{y} = - \int_{\mathcal{F}} \llbracket u \rrbracket(\mathbf{y}) \eta_h * \phi(\mathbf{y}) \, d\mathbf{y} = -(\llbracket u \rrbracket, \eta_h * \phi)_{\mathcal{F}}. \end{aligned} \tag{19}$$

On the other hand, according to [17], page 337, Exercise 10, $\eta_h * \llbracket u \rrbracket$ is locally integrable such that the statement of the lemma is valid for all bounded functions \mathbf{w} as it was stated. \square

Then we get as an obvious consequence the following.

Corollary 1 *The bilinear form a_η can be rewritten as*

$$\begin{aligned} a_\eta(u, v) &= (\eta_h * \nabla_h u, \eta_h * \nabla_h v) + \langle \eta_h * \nabla_h u, \eta_h * \llbracket v \rrbracket_{\mathcal{D}} \rangle + \langle \eta_h * \nabla_h v, \eta_h * \llbracket u \rrbracket_{\mathcal{D}} \rangle + (\eta_h * \llbracket u \rrbracket, \eta_h * \llbracket v \rrbracket) \\ &= (\eta_h * \nabla_h u, \eta_h * \nabla_h v) - (\eta_h * \eta_h * \nabla_h u, \llbracket v \rrbracket)_{\mathcal{F}} - (\eta_h * \eta_h * \nabla_h v, \llbracket u \rrbracket)_{\mathcal{F}} + (\eta_h * \llbracket u \rrbracket, \eta_h * \llbracket v \rrbracket). \end{aligned} \quad (20)$$

Note that the first line is related to the lifted forms of the dG methods as each scalar product corresponds to a volume integral. On the other hand, the second and third terms in the second line are integrals which can be computed on faces according to the second line in (19).

4 Comparison with the IP bilinear form

We compare our bilinear form (20) with the IP bilinear form (2) componentwise.

The first lemma quantifies the difference of the first terms.

Lemma 3 *For all $u, v \in \mathbb{P}_{h,\mathbf{k}}(\Omega)$ we have*

$$|(\nabla_h u, \nabla_h v) - (\eta_h * \nabla_h u, \eta_h * \nabla_h v)| \leq h^{s-1} \|\eta_h * \nabla_h u\| \|\eta_h * \nabla_h v\|.$$

Proof: We obviously have

$$\begin{aligned} &|(\nabla_h u, \nabla_h v) - (\eta_h * \nabla_h u, \eta_h * \nabla_h v)| \\ &\leq |(\nabla_h u - \eta_h * \nabla_h u, \nabla_h v) + (\eta_h * \nabla_h u, \nabla_h v - \eta_h * \nabla_h v)| \\ &\leq \|\nabla_h u - \eta_h * \nabla_h u\| \|\nabla_h v\| + \|\eta_h * \nabla_h u\| \|\nabla_h v - \eta_h * \nabla_h v\|. \end{aligned} \quad (21)$$

Also, application of the estimate in Proposition 2 and a simple scaling argument implies for each subdomain Ω_j that

$$\begin{aligned} \|\nabla_h u|_{\Omega_{j0}}\| &\leq \|\nabla_h u - \eta_h * \nabla_h u\|_{\Omega_{j0}} + \|\eta_h * \nabla_h u\|_{\Omega_{j0}} \\ &\leq h^{\frac{s-1}{2}} \|\nabla_h u\|_{\Omega_j} + \|\eta_h * \nabla_h u\|_{\Omega_{j0}} \lesssim h^{\frac{s-1}{2}} \|\nabla_h u\|_{\Omega_{j0}} + \|\eta_h * \nabla_h u\|_{\Omega_{j0}} \end{aligned}$$

and therefore,

$$\|\nabla_h u\|_{\Omega_{j0}} \lesssim \|\eta_h * \nabla_h u\|_{\Omega_{j0}},$$

which can be used to obtain the following inequality:

$$\|\nabla_h u\|_{\Omega_j} \lesssim \|\nabla_h u\|_{\Omega_{j0}} \lesssim \|\eta_h * \nabla_h u\|_{\Omega_{j0}} \leq \|\eta_h * \nabla u\|_{\Omega_j}. \quad (22)$$

Therefore, using again Proposition 2 we also have

$$\|\nabla_h u - \eta_h * \nabla_h u\|_{\Omega_j} \lesssim h^{\frac{s-1}{2}} \|\eta_h * \nabla u\|_{\Omega_j}. \quad (23)$$

Taking the square of (22) and (23) for each index j and summing them we have

$$\|\nabla_h u\| \lesssim \|\eta_h * \nabla u\| \quad \text{and} \quad \|\nabla_h u - \eta_h * \nabla_h u\| \lesssim h^{\frac{s-1}{2}} \|\eta_h * \nabla u\|$$

which can be used in (21) to obtain

$$\begin{aligned} &|(\nabla_h u, \nabla_h v) - (\eta_h * \nabla_h u, \eta_h * \nabla_h v)| \\ &\lesssim h^{\frac{s-1}{2}} \|\eta_h * \nabla u\| \|\eta_h * \nabla v\| + h^{\frac{s-1}{2}} \|\eta_h * \nabla v\| \|\eta_h * \nabla u\| \end{aligned}$$

as stated in the lemma. \square

To compare the second and third terms in (20) and (2) we use the notation in Fig. 1 and the corresponding explanation.

To analyze the average of the approximations we use the following statement on integral means.

Proposition 4 For each $u \in \mathbb{P}_{h,\mathbf{k}}(\Omega_- \cup \Omega_+)$ and $\mathbf{x} \in f_\Omega$ with $B(\mathbf{x}, 2h^s) \subset \Omega_- \cup \Omega_+$ there exist $\bar{\mathbf{x}}_- \in B_-(\mathbf{x}, 2h^s)$ and $\bar{\mathbf{x}}_+ \in B_+(\mathbf{x}, 2h^s)$ such that

$$u(\bar{\mathbf{x}}_-) = 2 \int_{B_-(\mathbf{0}, 2h^s)} u(\mathbf{x} - \mathbf{z}) \cdot \eta_h * \eta_h(\mathbf{z}) \, d\mathbf{z}$$

and similarly,

$$u(\bar{\mathbf{x}}_+) = 2 \int_{B_+(\mathbf{0}, 2h^s)} u(\mathbf{x} - \mathbf{z}) \cdot \eta_h * \eta_h(\mathbf{z}) \, d\mathbf{z},$$

where $B_-(\mathbf{0}, 2h^s)$ and $B_+(\mathbf{0}, 2h^s)$ denote the half-ball with non-positive and non-negative first coordinates, respectively.

The proof is postponed to the Appendix.

Proposition 5 For all $u \in \mathbb{P}_{h,\mathbf{k}}(\Omega_- \cup \Omega_+)$ we have the following inequality:

$$\max_{\substack{\mathbf{x} \in f_\Omega \\ B(\mathbf{x}, 2h^s) \subset \Omega_+ \cup \bar{\Omega}_-}} |\eta_h * \eta_h * \nabla_h u(\mathbf{x}) - \{\!\{ \nabla_h u \}\!\}(\mathbf{x})| \lesssim h^{s-\frac{d}{2}-1} \|\nabla_h u\|_{B(\Omega_+ \cup \Omega_-)}. \quad (24)$$

Proof: Using the result of Proposition 4 we rewrite the difference on the left hand side of (24) as follows:

$$\begin{aligned} & \eta_h * \eta_h * \nabla_h u(\mathbf{x}) - \{\!\{ \nabla_h u \}\!\}(\mathbf{x}) \\ &= \frac{1}{2} \left(2 \int_{B_-(\mathbf{0}, 2h^s)} \nabla_h u(\mathbf{x} - \mathbf{z}) \cdot \eta_h * \eta_h(\mathbf{z}) \, d\mathbf{z} + 2 \int_{B_+(\mathbf{0}, 2h^s)} \nabla_h u(\mathbf{x} - \mathbf{z}) \cdot \eta_h * \eta_h(\mathbf{z}) \, d\mathbf{z} \right. \\ & \quad \left. - \nabla_h u(\mathbf{x}_-) - \nabla_h u(\mathbf{x}_+) \right) \\ &= \frac{1}{2} (\nabla_h u(\bar{\mathbf{x}}_-) - \nabla_h u(\mathbf{x}_-) + \nabla_h u(\bar{\mathbf{x}}_+) - \nabla_h u(\mathbf{x}_+)). \end{aligned} \quad (25)$$

We use then the estimate

$$|\nabla_h u(\bar{\mathbf{x}}_-) - \nabla_h u(\mathbf{x}_-)| \leq 2h^s \cdot \sup_{\mathbf{z} \in (\mathbf{x}_-, (0-, \mathbf{y}))} \|\nabla_h^2 u(\mathbf{z})\|$$

in (25) to see that

$$\begin{aligned} & \max_{\substack{\mathbf{x} \in f_\Omega \\ B(\mathbf{x}, 2h^s) \subset \Omega_+ \cup \bar{\Omega}_-}} |\eta_h * \eta_h * \nabla_h u(\mathbf{x}) - \{\!\{ \nabla_h u \}\!\}(\mathbf{x})| \\ & \leq \frac{1}{2} \cdot 2h^s \cdot \left(\max_{\mathbf{z} \in f_\Omega + B(\mathbf{0}, 2h^s)} \|\nabla_h^2 u(\mathbf{z})\| + \max_{\mathbf{z} \in f_\Omega + B(\mathbf{0}, 2h^s)} \|\nabla_h^2 u(\mathbf{z})\| \right) \\ & = 2h^s \max_{\mathbf{z} \in f_\Omega + B(\mathbf{0}, 2h^s)} \|\nabla_h^2 u(\mathbf{z})\|. \end{aligned}$$

The last term here can be estimated using scaling arguments as

$$\begin{aligned} & 2h^s \max_{\mathbf{z} \in f_\Omega + B(\mathbf{0}, 2h^s)} \|\nabla_h^2 u(\mathbf{z})\| \lesssim h^s \sqrt{h^{-s} h_\Omega^{1-d}} \|\nabla_h^2 u\|_{f_\Omega + B(\mathbf{0}, 2h^s)} \\ & \lesssim h^{\frac{1+s-d}{2}} \|\nabla_h^2 u\|_{f_\Omega + B(\mathbf{0}, 2h^s)} \lesssim h^{\frac{1+s-d}{2}} h^{\frac{s-1}{2}} \|\nabla_h^2 u\|_{\Omega_+ \cup \Omega_-} \lesssim h^{s-\frac{d}{2}-1} \|\nabla_h u\|_{\Omega_+ \cup \Omega_-}, \end{aligned}$$

which proves the statement of the proposition. \square

We can now relate the third and second terms in the proposed bilinear form (20) and the IP bilinear form.

Lemma 4 For arbitrary $u, v \in \mathbb{P}_{h,k}$ we have the following inequality:

$$|(\eta_h * \eta_h * \nabla_h u, \llbracket v \rrbracket)_{\mathcal{F}} - (\{\nabla_h u\}, \llbracket v \rrbracket)_{\mathcal{F}}| \lesssim h^{s-1} \|\nabla(\eta_h * u)\| \|\nabla(\eta_h * v)\|.$$

Proof: Using the result of Proposition 5 and Proposition 3 we obtain the following estimation on the interelement face f_Ω between Ω_+ and Ω_- :

$$\begin{aligned} & |(\eta_h * \eta_h * \nabla_h u, \llbracket v \rrbracket)_{f_\Omega} - (\{\nabla_h u\}, \llbracket v \rrbracket)_{f_\Omega}| = |(\eta_h * \eta_h * \nabla_h u - \{\nabla_h u\}, \llbracket v \rrbracket)_{f_\Omega}| \\ & \leq \max_{\mathbf{x} \in f_\Omega} |(\eta_h * \eta_h * \nabla_h u - \{\nabla_h u\})(\mathbf{x})| \int_{f_\Omega} |\llbracket v \rrbracket| \\ & \lesssim \max_{\substack{\mathbf{x} \in f_\Omega \\ B(\mathbf{x}, 2h^s) \subset \Omega_+ \cup \Omega_-}} |(\eta_h * \eta_h * \nabla_h u - \{\nabla_h u\})(\mathbf{x})| \int_{f_\Omega} |\llbracket v \rrbracket| \\ & \leq h^{s-\frac{d}{2}-1} \left(\frac{h}{h_\Omega}\right)^{\frac{d}{2}} \|\nabla_h u\|_{\Omega_+ \cup \Omega_-} \int_f |\llbracket v \rrbracket| \leq h^{s-\frac{d}{2}-1} \left(\frac{h}{h_\Omega}\right)^{\frac{d}{2}} \|\nabla_h u\|_{\Omega_+ \cup \Omega_-} \int_{\Omega_+ \cup \Omega_-} |\nabla(\eta_h * v)| \\ & \lesssim h^{s-\frac{d}{2}-1} \left(\frac{h}{h_\Omega}\right)^{\frac{d}{2}} \|\nabla_h(\eta_h * u)\|_{\Omega_+ \cup \Omega_-} h_\Omega^{\frac{d}{2}} \|\nabla(\eta_h * v)\|_{\Omega_+ \cup \Omega_-} \\ & \lesssim h^{s-1} \|\nabla_h(\eta_h * u)\|_{\Omega_+ \cup \Omega_-} \|\nabla(\eta_h * v)\|_{\Omega_+ \cup \Omega_-}. \end{aligned}$$

Summing up these inequalities for each interelement face f_Ω and using the discrete Cauchy–Schwarz inequality result in the estimate

$$\begin{aligned} & \left| \sum_{f_\Omega \in \mathcal{F}} (\eta_h * \eta_h * \nabla_h u, \llbracket v \rrbracket)_{f_\Omega} - \sum_{f_\Omega \in \mathcal{F}} (\{\nabla_h u\}, \llbracket v \rrbracket)_{f_\Omega} \right| \leq \sum_{f_\Omega \in \mathcal{F}} |(\eta_h * \eta_h * \nabla_h u, \llbracket v \rrbracket)_{f_\Omega} - (\{\nabla_h u\}, \llbracket v \rrbracket)_{f_\Omega}| \\ & \leq \sum_{f_\Omega \in \mathcal{F}} h^{s-1} \|\nabla_h(\eta_h * u)\|_{\Omega_+ \cup \Omega_-} \|\nabla_h(\eta_h * v)\|_{\Omega_+ \cup \Omega_-} \\ & \lesssim h^{s-1} \sqrt{\sum_{f_\Omega \in \mathcal{F}} \|\nabla(\eta_h * u)\|_{\Omega_+ \cup \Omega_-}^2} \sqrt{\sum_{f_\Omega \in \mathcal{F}} \|\nabla(\eta_h * v)\|_{\Omega_+ \cup \Omega_-}^2} \lesssim h^{s-1} \|\nabla(\eta_h * u)\|^2 \|\nabla(\eta_h * v)\|^2 \end{aligned}$$

as stated in the lemma. \square

To relate the last term in (20) with the penalty term in the IP bilinear form, we rewrite the locally integrable function $\eta_h * \llbracket v \rrbracket$ (see Lemma 2) in a more explicit form.

Lemma 5 For each $v \in \mathbb{P}_{h,k}$ and $f \in \mathcal{F}$ the following identity is valid:

$$\eta_h * \llbracket v \rrbracket_f(\mathbf{x}) = \int_f \eta_h(\mathbf{x} - \mathbf{y}) \llbracket v \rrbracket_f(\mathbf{y}) \, d\mathbf{y}. \quad (26)$$

This result can also serve as a good argument why did we apply the same notation for the convolution corresponding to the jump of v and the jump function. Since the proof is a bit technical it is postponed to the appendix.

To analyze the right hand side of (26), we introduce the following sets which are depicted in Figure 2.

$$f \otimes r = \{\mathbf{x} \in \langle f \rangle \dot{+} r\boldsymbol{\nu}_1 \cup \langle f \rangle \dot{+} r\boldsymbol{\nu}_2 : d(\mathbf{x}, f) \leq h^s\}$$

and

$$f_0 \otimes r = (f_0 \dot{+} r\boldsymbol{\nu}_1) \cup (f_0 \dot{+} r\boldsymbol{\nu}_2).$$

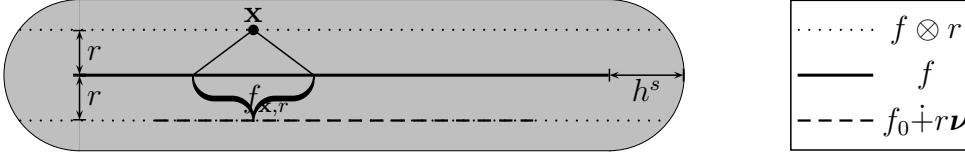


Figure 2: Interelement face f with the support of $\eta_h * [u]_f$ (shaded) and the sets $f_{\mathbf{x},r}$, $f \otimes r$ and $f_0 + r\nu$.

, where $\langle f \rangle$ denotes the affine subspace generated by f .

Observe that $\eta_h * [u]_f(\mathbf{x})$ can be nonzero if $\mathbf{x} \in f \otimes r$ for some $r < h^s$ and then we use the notation $f_{\mathbf{x},r} = B(\mathbf{x}, h^s) \cap f$, which is a ball in $\langle f \rangle$ centered at the projection of \mathbf{x} on f with the radius $\sqrt{h^{2s} - r^2}$ such that $\lambda(f_{\mathbf{x},r}) = B_{\sqrt{h^{2s} - r^2}, d-1}$.

With these, we can rewrite (26) as

$$\eta_h * [u]_f(\mathbf{x}) = \frac{1}{B_{h^s, d}} \int_{f_{\mathbf{x},r}} [u] \, ds.$$

In this way, using Lemma 5 the integral in the last term of (20) on a face f can be rewritten as

$$(\eta_h * [u]_f, \eta_h * [v]_f) = \frac{1}{[B_{h^s, d}]^2} \int_{-h^s}^{h^s} \int_{f \otimes r} \int_{f_{\mathbf{x},r}} [u](\mathbf{s}) \, ds \int_{f_{\mathbf{x},r}} [v](\mathbf{s}) \, ds \, d\mathbf{x} \, dr. \quad (27)$$

We intend to relate this term with the following:

$$\frac{1}{[B_{h^s, d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s} - r^2}, d-1}]^2 [u][v](\mathbf{x}) \, d\mathbf{x} \, dr. \quad (28)$$

To work with smooth functions, both in (27) and (28) we have to restrict the integrals on $f_0 \otimes r$ and to f_0 , respectively. Since $\lambda(f) \sim h^{d-1}$ and $\lambda(f \setminus f_0) \sim h^{d-2}h^s$, a scaling argument implies the following estimates:

$$\begin{aligned} I_1(r) &:= \left| \int_{f \otimes r} \int_{f_{\mathbf{x},r}} [u](\mathbf{s}) \, ds \int_{f_{\mathbf{x},r}} [v](\mathbf{s}) \, ds \, d\mathbf{x} - \int_{f_0} \int_{B(\mathbf{x}, \sqrt{h^{2s} - r^2})} [u](\mathbf{s}) \, ds \int_{B(\mathbf{x}, \sqrt{h^{2s} - r^2})} [v](\mathbf{s}) \, ds \, d\mathbf{x} \right| \\ &= \left| \int_{f \otimes r} \int_{f_{\mathbf{x},r}} [u](\mathbf{s}) \, ds \int_{f_{\mathbf{x},r}} [v](\mathbf{s}) \, ds \, d\mathbf{x} - \int_{f_0 \otimes r} \int_{f_{\mathbf{x},r}} [u](\mathbf{s}) \, ds \int_{f_{\mathbf{x},r}} [v](\mathbf{s}) \, ds \, d\mathbf{x} \right| \\ &\lesssim \frac{h^{d-2}h^s}{h^{d-1}} \int_{f \otimes r} \left| \int_{f_{\mathbf{x},r}} [u](\mathbf{s}) \, ds \right| \left| \int_{f_{\mathbf{x},r}} [v](\mathbf{s}) \, ds \right| \, d\mathbf{x} = h^{s-1} \int_{f \otimes r} \left| \int_{f_{\mathbf{x},r}} [u](\mathbf{s}) \, ds \right| \left| \int_{f_{\mathbf{x},r}} [v](\mathbf{s}) \, ds \right| \, d\mathbf{x} \end{aligned} \quad (29)$$

and

$$\begin{aligned} I_2(r) &:= \left| \int_{f_0} [B_{\sqrt{h^{2s} - r^2}, d-1}]^2 [u][v](\mathbf{x}) \, d\mathbf{x} - \int_f [B_{\sqrt{h^{2s} - r^2}, d-1}]^2 [u][v](\mathbf{x}) \, d\mathbf{x} \right| \\ &\lesssim \frac{h^{d-2}h^s}{h^{d-1}} \int_f [B_{\sqrt{h^{2s} - r^2}, d-1}]^2 |[u][v]|(\mathbf{x}) \, d\mathbf{x} \lesssim h^{s-1}(h^{2s} - r^2)^{d-1} \int_f |[u][v]|(\mathbf{x}) \, d\mathbf{x}. \end{aligned} \quad (30)$$

Remark: The estimation of I_1 is still valid if we use $f_{00} \subset f$ with $\lambda(f \setminus f_{00}) \sim h^{d-2}h^s$.

For the forthcoming computations, we also give the magnitude of the following integrals:

$$\int_{-h^s}^{h^s} (h^{2s} - r^2)^{d-1} dr = \mathcal{O}(h^{2sd-s}) \quad (31)$$

$$\int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} |\mathbf{s} - \mathbf{x}|^2 d\mathbf{s} = \mathcal{O}(\sqrt{h^{2s}-r^2}), \quad (32)$$

which can be verified with a straightforward computation.

Lemma 6 *For all $u, v \in \mathbb{P}_{h,\mathbf{k}}$ and $\Omega_+, \Omega_- \in \mathcal{T}_h$ we have the following inequality*

$$\begin{aligned} & \left| (\eta_h * \llbracket u \rrbracket_f, \eta_h * \llbracket v \rrbracket_f) - \frac{1}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2},d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) d\mathbf{x} dr \right| \\ & \leq h^{s-1} (1 + h^{3s-d-2}) \|\nabla(\eta_h * u)\|_{\Omega_+ \cup \Omega_-} \|\nabla(\eta_h * v)\|_{\Omega_+ \cup \Omega_-}. \end{aligned} \quad (33)$$

Proof: Using (27) and a triangle inequality with (29) and (30) we have

$$\begin{aligned} & \left| (\eta_h * \llbracket u \rrbracket_f, \eta_h * \llbracket v \rrbracket_f) - \frac{1}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2},d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) d\mathbf{x} dr \right| \\ & \leq \frac{1}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} I_1(r) dr + \frac{1}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} I_2(r) dr \\ & + \frac{1}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \left| \int_{f_0} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \llbracket u \rrbracket (\mathbf{s}) d\mathbf{s} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \llbracket v \rrbracket (\mathbf{s}) d\mathbf{s} d\mathbf{x} \right. \\ & \quad \left. - \int_{f_0} [B_{\sqrt{h^{2s}-r^2},d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) d\mathbf{x} \right| dr \\ & \leq \frac{h^{s-1}}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \int_{f \otimes r} \left| \int_{f_{\mathbf{x},r}} \llbracket u \rrbracket (\mathbf{s}) d\mathbf{s} \right| \left| \int_{f_{\mathbf{x},r}} \llbracket v \rrbracket (\mathbf{s}) d\mathbf{s} \right| d\mathbf{x} dr \\ & + \frac{h^{s-1}}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2},d-1}]^2 |\llbracket u \rrbracket \llbracket v \rrbracket| (\mathbf{x}) d\mathbf{x} dr \\ & + \frac{1}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \left| \int_{f_0} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \llbracket u \rrbracket (\mathbf{s}) d\mathbf{s} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \llbracket v \rrbracket (\mathbf{s}) d\mathbf{s} \right. \\ & \quad \left. - [B_{\sqrt{h^{2s}-r^2},d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) d\mathbf{x} \right| dr. \end{aligned} \quad (34)$$

The error terms here are estimated separately.

We first use (7) and (31) to obtain

$$\begin{aligned}
& \frac{h^{s-1}}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \int_{f \otimes r} \left| \int_{f_{\mathbf{x},r}} \llbracket u \rrbracket(\mathbf{s}) \, d\mathbf{s} \right| \left| \int_{f_{\mathbf{x},r}} \llbracket v \rrbracket(\mathbf{s}) \, d\mathbf{s} \right| \, d\mathbf{x} \, dr \\
& \lesssim h^{s-2sd-1} \int_{-h^s}^{h^s} \lambda_{d-1}(f \otimes r) [B_{\sqrt{h^{2s}-r^2},d-1}]^2 \max_f \llbracket u \rrbracket \max_f \llbracket v \rrbracket \, dr \\
& \lesssim h^{s-2sd-1} \int_{-h^s}^{h^s} h_{\Omega}^{d-1} (h^{2s} - r^2)^{d-1} \cdot h_{\Omega}^{1-d} \int_f |\llbracket u \rrbracket| h_{\Omega}^{1-d} \int_f |\llbracket v \rrbracket| \, dr \\
& \leq h^{s-2sd-d} \int_{-h^s}^{h^s} (h^{2s} - r^2)^{d-1} \, dr \int_f |\llbracket u \rrbracket| \int_f |\llbracket v \rrbracket| \\
& = h^{s-2sd-d} h^{2sd-s} \int_f |\llbracket u \rrbracket| \int_f |\llbracket v \rrbracket| = h^{-d} \int_f |\llbracket u \rrbracket| \int_f |\llbracket v \rrbracket|. \tag{35}
\end{aligned}$$

We proceed similarly for the second term in (34):

$$\begin{aligned}
& \frac{h^{s-1}}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2},d-1}]^2 |\llbracket u \rrbracket(\mathbf{x}) \llbracket v \rrbracket(\mathbf{x})| \, d\mathbf{x} \, dr \\
& \lesssim h^{s-2sd-1} h^{2sd-s} \int_f |\llbracket u \rrbracket \llbracket v \rrbracket| \lesssim h^{-1} h_{\Omega}^{1-d} \int_f |\llbracket u \rrbracket| \int_f |\llbracket v \rrbracket| \leq h^{-d} \int_f |\llbracket u \rrbracket| \int_f |\llbracket v \rrbracket|.
\end{aligned}$$

We finally estimate the third term in (34). Using the expansion in (11) on f_0 with the surface gradient $\nabla_f \llbracket u \rrbracket := \nabla \llbracket u \rrbracket$ and integrating both sides on the ball $B(\mathbf{x}, \sqrt{h^{2s}-r^2})$ implies

$$\int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \llbracket u \rrbracket(\mathbf{s}) \, d\mathbf{s} = B_{\sqrt{h^{2s}-r^2},d-1} \llbracket u \rrbracket(\mathbf{x}) + \frac{1}{2} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \nabla^2 \llbracket u \rrbracket(\boldsymbol{\xi}_{\mathbf{s}}) (\mathbf{s} - \mathbf{x}) \cdot (\mathbf{s} - \mathbf{x}) \, d\mathbf{s}. \tag{36}$$

Taking the product of (36) for $\llbracket u \rrbracket$ and $\llbracket v \rrbracket$ and using (32) and (9) we obtain

$$\begin{aligned}
& \left| \int_{f_0} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \llbracket u \rrbracket(\mathbf{s}) \, d\mathbf{s} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \llbracket v \rrbracket(\mathbf{s}) \, d\mathbf{s} \, d\mathbf{x} - [B_{\sqrt{h^{2s}-r^2}, d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket(\mathbf{x}) \, d\mathbf{x} \right| \\
& \leq \int_{f_0} \left| B_{\sqrt{h^{2s}-r^2}, d-1} \llbracket u \rrbracket(\mathbf{x}) \frac{1}{2} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \nabla^2 \llbracket v \rrbracket(\boldsymbol{\xi}_s) |\mathbf{s} - \mathbf{x}|^2 \, d\mathbf{s} \right| \\
& + \left| B_{\sqrt{h^{2s}-r^2}, d-1} \llbracket v \rrbracket(\mathbf{x}) \frac{1}{2} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \nabla^2 \llbracket u \rrbracket(\boldsymbol{\xi}_s) |\mathbf{s} - \mathbf{x}|^2 \, d\mathbf{s} \right| \\
& + \left| \frac{1}{4} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \nabla^2 \llbracket u \rrbracket(\boldsymbol{\xi}_s) |\mathbf{s} - \mathbf{x}|^2 \, d\mathbf{s} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \nabla^2 \llbracket v \rrbracket(\boldsymbol{\xi}_s) |\mathbf{s} - \mathbf{x}|^2 \, d\mathbf{s} \right| \, d\mathbf{x} \\
& \lesssim \max_f |\nabla^2 \llbracket v \rrbracket| B_{\sqrt{h^{2s}-r^2}, d-1} \int_f |\llbracket u \rrbracket|(\mathbf{x}) \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} |\mathbf{s} - \mathbf{x}|^2 \, d\mathbf{s} \, d\mathbf{x} \\
& + \max_f |\nabla^2 \llbracket u \rrbracket| B_{\sqrt{h^{2s}-r^2}, d-1} \int_f |\llbracket v \rrbracket|(\mathbf{x}) \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} |\mathbf{s} - \mathbf{x}|^2 \, d\mathbf{s} \, d\mathbf{x} \\
& + \max_f |\nabla^2 \llbracket v \rrbracket| \max_f |\nabla^2 \llbracket u \rrbracket| \int_{f_0} (h^{2s} - r^2)^{d+1} \, d\mathbf{x} \\
& \lesssim h^{-d-1} \int_f |\llbracket v \rrbracket| B_{\sqrt{h^{2s}-r^2}, d-1} \int_f |\llbracket u \rrbracket| (h^{2s} - r^2)^{\frac{d+1}{2}} + h^{-2d-2} \int_f |\llbracket v \rrbracket| \int_f |\llbracket u \rrbracket| (h^{2s} - r^2)^{d+1} \\
& \lesssim h^{-d-1} (h^{2s} - r^2)^d \int_f |\llbracket v \rrbracket| \int_f |\llbracket u \rrbracket| + h^{-2d-2} (h^{2s} - r^2)^{d+1} \int_f |\llbracket v \rrbracket| \int_f |\llbracket u \rrbracket| \\
& = (h^{-d-1} (h^{2s} - r^2)^d + h^{-2d-2} (h^{2s} - r^2)^{d+1}) \int_f |\llbracket v \rrbracket| \int_f |\llbracket u \rrbracket|.
\end{aligned}$$

In this way, we can estimate the last term in (34) as

$$\begin{aligned}
& \frac{1}{[B_{h^s, d}]^2} \int_{-h^s}^{h^s} \left| \int_{f_0} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \llbracket u \rrbracket(\mathbf{s}) \, d\mathbf{s} \int_{B(\mathbf{x}, \sqrt{h^{2s}-r^2})} \llbracket v \rrbracket(\mathbf{s}) \, d\mathbf{s} - [B_{\sqrt{h^{2s}-r^2}, d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket(\mathbf{x}) \, d\mathbf{x} \right| \, dr \\
& \lesssim \frac{1}{[B_{h^s, d}]^2} \int_{-h^s}^{h^s} (h^{-d-1} (h^{2s} - r^2)^d + h^{-2d-2} (h^{2s} - r^2)^{d+1}) \int_f |\llbracket v \rrbracket| \int_f |\llbracket u \rrbracket| \, dr \\
& \lesssim h^{-2sd} \int_f |\llbracket v \rrbracket| \int_f |\llbracket u \rrbracket| \int_{-h^s}^{h^s} h^{-d-1} (h^{2s} - r^2)^d + h^{-2d-2} (h^{2s} - r^2)^{d+1} \, dr \\
& \lesssim h^{-2sd} \int_f |\llbracket v \rrbracket| \int_f |\llbracket u \rrbracket| \cdot (h^{-d-1} h^{2sd+s} + h^{-2d-2} h^{2sd+3s}) \\
& = (h^{-d-1+s} + h^{-2d-2+3s}) \int_f |\llbracket v \rrbracket| \int_f |\llbracket u \rrbracket|
\end{aligned}$$

and therefore, using (34) and the estimate (18) in Proposition 3 we finally obtain

$$\begin{aligned}
& \left| (\eta_h * \llbracket u \rrbracket_f, \eta_h * \llbracket v \rrbracket_f) - \frac{1}{[B_{h^s, d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2}, d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket(\mathbf{x}) \, d\mathbf{x} \, dr \right| \\
& \lesssim (h^{-d} + h^{-d-1+s} + h^{-2d-2+3s}) \int_f |\llbracket v \rrbracket| \int_f |\llbracket u \rrbracket| \\
& \lesssim (h^{-d} + h^{-d-1+s} + h^{-2d-2+3s}) h^d h^{s-1} \|\nabla(\eta_h * u)\|_\Omega \|\nabla(\eta_h * v)\|_\Omega \\
& \lesssim h^{s-1} (1 + h^{-d+3s-2}) \|\nabla(\eta_h * u)\|_\Omega \|\nabla(\eta_h * v)\|_\Omega
\end{aligned}$$

as we have stated. \square

Corollary 2 *For all $u, v \in \mathbb{P}_{h,\mathbf{k}}$ we have*

$$\left| \sum_{f \in \mathcal{F}} (\eta_h * \llbracket u \rrbracket, \eta_h * \llbracket v \rrbracket)_f - \sum_{f \in \mathcal{F}} \frac{1}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2},d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) \, d\mathbf{x} \, dr \right| \leq h^{s-1} (1 + h^{3s-d-2}) \|\nabla(\eta_h * u)\| \|\nabla(\eta_h * v)\|.$$

Taking the sum of the inequalities in (33) and applying the discrete Cauchy–Schwarz inequality $|\sum_{j \in J} a_j b_j| \leq \sqrt{\sum_{j \in J} a_j} \sqrt{\sum_{j \in J} b_j}$ we obtain

$$\begin{aligned} & \left| \sum_{f \in \mathcal{F}} (\eta_h * \llbracket u \rrbracket, \eta_h * \llbracket v \rrbracket)_f - \sum_{f \in \mathcal{F}} \frac{1}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2},d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) \, d\mathbf{x} \, dr \right| \\ & \leq \sum_{f \in \mathcal{F}} \left| (\eta_h * \llbracket u \rrbracket, \eta_h * \llbracket v \rrbracket)_f - \frac{1}{[B_{h^s,d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2},d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) \, d\mathbf{x} \, dr \right| \\ & \leq h^{s-1} (1 + h^{3s-d-2}) \sum_{f \in \mathcal{F}} \|\nabla(\eta_h * u)\|_{\Omega_+ \cup \Omega_-} \|\nabla(\eta_h * v)\|_{\Omega_+ \cup \Omega_-} \\ & \lesssim h^{s-1} (1 + h^{3s-d-2}) \|\nabla(\eta_h * u)\| \|\nabla(\eta_h * v)\| \end{aligned}$$

as stated in the corollary. \square

Remark: The above difference is lower order compared to $\|\nabla(\eta_h * u)\| \|\nabla(\eta_h * v)\|$ provided that $4s - d - 3 > 0$ which is ensured for $s > 1.5$.

Finally, we compute the approximation of the penalty term in (28), which appears in Lemma 6.

- For $d = 2$ we have

$$\begin{aligned} & \frac{1}{[B_{h^s,2}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2},1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) \, d\mathbf{x} \, dr \\ & = \frac{1}{h^{4s}\pi^2} \int_{-h^s}^{h^s} 4(h^{2s} - r^2) \, dr \int_f \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) \, d\mathbf{x} = \frac{16}{3\pi^2} h^{-s} \int_f \llbracket u \rrbracket \llbracket v \rrbracket. \end{aligned}$$

- For $d = 3$ we have

$$\begin{aligned} & \frac{1}{[B_{h^s,3}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2},2}]^2 \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) \, d\mathbf{x} \, dr \\ & = \frac{9}{16h^{6s}\pi^2} \int_{-h^s}^{h^s} \pi^2 (h^{2s} - r^2)^2 \, dr \int_f \llbracket u \rrbracket \llbracket v \rrbracket (\mathbf{x}) \, d\mathbf{x} = \frac{3}{5} h^{-s} \int_f \llbracket u \rrbracket \llbracket v \rrbracket. \end{aligned}$$

To prove the first main result we introduce the IP bilinear form $a_{\text{IP},s} : \mathbb{P}_{h,\mathbf{k}} \times \mathbb{P}_{h,\mathbf{k}} \rightarrow \mathbb{R}$ with

$$a_{\text{IP},s}(u, v) = (\nabla_h u, \nabla_h v) - \sum_{f \in \mathcal{F}} (\{\nabla_h u\}, \llbracket v \rrbracket)_f + (\{\nabla_h v\}, \llbracket u \rrbracket)_f + \sum_{f \in \mathcal{F}} \sigma_{s,h}(\llbracket u \rrbracket, \llbracket v \rrbracket)_f, \quad (37)$$

where

$$\sigma_{s,h}(\llbracket u \rrbracket, \llbracket v \rrbracket)_f = \begin{cases} \frac{16}{3\pi^2} h^{-s} (\llbracket u \rrbracket, \llbracket v \rrbracket)_f & \text{for } d = 2 \\ \frac{3}{5} h^{-s} (\llbracket u \rrbracket, \llbracket v \rrbracket)_f & \text{for } d = 3. \end{cases}$$

and the corresponding finite element approximation $u_{\text{IP},s}$ for which

$$a_{\text{IP},s}(u_{\text{IP},s}, v) = (g, v) \quad \forall v \in \mathbb{P}_{h,\mathbf{k}}. \quad (38)$$

Remark: Since we have the restriction $s > 1.5$, the bilinear form $a_{\text{IP},s}$ can be recognized as an overpenalized IP bilinear form.

Theorem 1 *Assume that $3s > d+2$. Then the IP bilinear form in (37) is a lower-order perturbation of a_η in the sense that*

$$|a_\eta(u, v) - a_{\text{IP},s}(u, v)| \lesssim h^{s-1}(1 + h^{3s-d-2}) \|\nabla(\eta_h * u)\| \|\nabla(\eta_h * v)\|.$$

Proof: Using Lemma 3, Lemma 4 and Corollary 2 we obtain

$$\begin{aligned} |a_\eta(u, v) - a_{\text{IP},s}(u, v)| &\leq |(\eta_h * \nabla_h u, \eta_h * \nabla_h v) - (\nabla_h u, \nabla_h v)| \\ &+ \left| \sum_{f \in \mathcal{F}} (\eta_h * \eta_h * \nabla_h u, \llbracket v \rrbracket)_f + (\eta_h * \eta_h * \nabla_h v, \llbracket u \rrbracket)_f - (\llbracket \nabla_h u \rrbracket, \llbracket v \rrbracket)_f - (\llbracket \nabla_h v \rrbracket, \llbracket u \rrbracket)_f \right| \\ &+ \left| \sum_{f \in \mathcal{F}} (\eta_h * \llbracket u \rrbracket, \eta_h * \llbracket v \rrbracket)_f - \sum_{f \in \mathcal{F}} \frac{1}{[B_{h^s, d}]^2} \int_{-h^s}^{h^s} \int_f [B_{\sqrt{h^{2s}-r^2}, d-1}]^2 \llbracket u \rrbracket \llbracket v \rrbracket(\mathbf{x}) \, d\mathbf{x} \, dr \right| \\ &\lesssim (h^{s-1} + h^{s-1}(1 + h^{3s-d-2})) \|\nabla(\eta_h * u)\| \|\nabla(\eta_h * v)\| \end{aligned}$$

as stated in the theorem. \square

Since the bilinear form a_η is a slight modification of $a_{\text{IP},s}$ we expect that the local average of the approximations of u_h and $u_{\text{IP},s}$ are also close to each other. In precise terms we have the following.

Theorem 2 *Assume that $3s > d+2$. Then for the finite element approximations u_h and $u_{\text{IP},s}$ we have*

$$\|\nabla(\eta_h * u_{\text{IP},s} - \eta_h * u_h)\| \lesssim h^{s-1} \|\nabla(\eta_h * u_h)\| + \max_j h_{\Omega_j}^d \|\eta_h * g - g_0\|.$$

Proof: Since u_h solves (5) and $u_{\text{IP},s} \in \mathbb{P}_{h,\mathbf{k}}$ we have

$$(\nabla(\eta_h * u_h), \nabla(\eta_h * (u_h - u_{\text{IP},s}))) = (g_0, \eta_h * (u_h - u_{\text{IP},s}))$$

such that using the equality

$$(\eta_h * w_1, w_2) = (w_1, \eta_h * w_2)$$

for compactly supported functions $w_1, w_2 \in L_1(\mathbb{R}^d)$ and the definition of $u_{\text{IP},s}$ in (38) we obtain

$$\begin{aligned} &(\nabla(\eta_h * (u_h - u_{\text{IP},s})), \nabla(\eta_h * (u_h - u_{\text{IP},s}))) \\ &= (g_0, \eta_h * (u_h - u_{\text{IP},s})) - (\nabla(\eta_h * u_{\text{IP},s}), \nabla(\eta_h * (u_h - u_{\text{IP},s}))) \\ &= (g_0, \eta_h * (u_h - u_{\text{IP},s})) - a_{\text{IP},s}(u_{\text{IP},s}, u_h - u_{\text{IP},s}) - (\nabla(\eta_h * u_{\text{IP},s}), \nabla(\eta_h * (u_h - u_{\text{IP},s}))) \\ &+ a_{\text{IP},s}(u_{\text{IP},s}, u_h - u_{\text{IP},s}) \\ &= (\eta_h * g, u_h - u_{\text{IP},s}) - (g, u_h - u_{\text{IP},s}) - (\nabla(\eta_h * (u_{\text{IP},s} - u_h)), \nabla(\eta_h * (u_h - u_{\text{IP},s}))) \\ &+ a_{\text{IP},s}(u_{\text{IP},s} - u_h, u_h - u_{\text{IP},s}) - (\nabla(\eta_h * u_h), \nabla(\eta_h * (u_h - u_{\text{IP},s}))) + a_{\text{IP},s}(u_h, u_h - u_{\text{IP},s}). \end{aligned} \quad (39)$$

We note that the application of (22) to $u_h - u_{\text{IP},s}$ (instead of $\nabla_h u$) and the Friedrichs's inequality imply

$$\|u_h - u_{\text{IP}}\| \lesssim \|\eta_h * (u_h - u_{\text{IP}})\| \lesssim \max_j h_{\Omega_j}^d \|\nabla(\eta_h * (u_h - u_{\text{IP}}))\|$$

and therefore, using Theorem 1 for the last two pair of terms in (39), we obtain that

$$\begin{aligned}
& \|\nabla(\eta_h * (u_h - u_{\text{IP},s}))\|^2 \\
& \lesssim \|\eta_h * g - g\| \|u_h - u_{\text{IP},s}\| + h^{s-1} (\|\nabla(\eta_h * (u_{\text{IP},s} - u_h))\|^2 + \|\nabla(\eta_h * u_h)\| \|\nabla(\eta_h * (u_h - u_{\text{IP},s}))\|) \\
& \lesssim \max_j h_{\Omega_j}^d \|\eta_h * g - g\| \|\nabla(\eta_h * (u_h - u_{\text{IP},s}))\| + h^{s-1} (1 + h^{3s-d-2}) \|\nabla(\eta_h * (u_{\text{IP},s} - u_h))\|^2 \\
& + h^{s-1} (1 + h^{3s-d-2}) \|\nabla(\eta_h * u_h)\| \|\nabla(\eta_h * (u_h - u_{\text{IP},s}))\|
\end{aligned}$$

such that we finally get

$$(1 - h^{s-1}) \|\nabla(\eta_h * (u_h - u_{\text{IP},s}))\| \lesssim \max_j h_{\Omega_j}^d \|\eta_h * g - g\| + h^{s-1} (1 + h^{3s-d-2}) \|\nabla(\eta_h * u_h)\|,$$

which implies the estimate in the theorem. \square

To state quasi optimal convergence we observe that for each h we have $\eta_h * u_h \in \mathbb{P}_{h,\mathbf{k},s} \subset H_0^1(\Omega_h)$. This means that the method in (5) is not conforming since the approximation in general is not in $H_0^1(\Omega)$. Also, the bilinear form a_η^+ is non-consistent in the sense that the zero extension u_0 of u is not necessarily the solution of (5) for any h . In this way, we need to apply the Strang lemma [11], Section 2.3.2. For this we first note that for some constants C_1 and C_2 we have for all $0 \neq w_1, w_2 \in H_0^1(\Omega_h)$ that

$$|a_\eta^+(w_1, w_2)| \leq C_1 \|w_1\|_{H_0^1(\Omega_h)} \|w_2\|_{H_0^1(\Omega_h)}$$

and

$$C_2 \leq \frac{a_\eta^+(w_1, w_2)}{\|w_1\|_{H_0^1(\Omega_h)} \|w_2\|_{H_0^1(\Omega_h)}}.$$

Lemma 7 *The numerical solution $\eta_h * u_h$ of (5) approximates u in quasi optimal way in the sense that*

$$\|\nabla(u - \eta_h * u_h)\|_\Omega \lesssim \inf_{v_h \in \mathbb{P}_{h,\mathbf{k}}} \|\nabla(u - \eta_h * v_h)\| + h^{s-\frac{1}{2}} \|\nabla u\|.$$

Proof: Since in this proof it is essential whether a scalar product is defined on Ω or on Ω_h , we indicate it in the subscript. A direct application of Lemma 2.25 in [11] gives that

$$\begin{aligned}
& \|\nabla(u - \eta_h * u_h)\|_{\Omega_h} \\
& \leq (1 + \frac{C_1}{C_2}) \inf_{v_h \in \mathbb{P}_{h,\mathbf{k}}} \|\nabla(u - \eta_h * v_h)\| + \sup_{\eta_h * v_h \in \mathbb{P}_{h,\mathbf{k},s}} \frac{(\nabla u_0, \nabla(\eta_h * v_h))_{\Omega_h} - (g_0, \eta_h * v_h)_{\Omega_h}}{\|\eta_h * v_h\|_{1,\Omega_h}}, \tag{40}
\end{aligned}$$

where the lower indices denote zero extensions. Using these, the second term in (40) can be rewritten as

$$\begin{aligned}
& (\nabla u_0, \nabla(\eta_h * v_h))_{\Omega_h} - (g_0, \eta_h * v_h)_{\Omega_h} = (\nabla u, \nabla(\eta_h * v_h))_\Omega - (g, \eta_h * v_h)_\Omega \\
& = (-\Delta u, \eta_h * v_h)_\Omega + \langle \boldsymbol{\nu} \cdot \nabla u, \eta_h * v_h \rangle_{\partial\Omega} - (g, \eta_h * v_h)_\Omega = \langle \boldsymbol{\nu} \cdot \nabla u, \eta_h * v_h \rangle_{\partial\Omega}.
\end{aligned}$$

Therefore, we can estimate (40) to obtain

$$\|\nabla(u - \eta_h * u_h)\|_\Omega \lesssim \inf_{v_h \in \mathbb{P}_{h,\mathbf{k}}} \|\nabla(u - \eta_h * v_h)\| + \sup_{\eta_h * v_h \in \mathbb{P}_{h,\mathbf{k},s}} \frac{\langle \boldsymbol{\nu} \cdot \nabla u, \eta_h * v_h \rangle_{\partial\Omega}}{\|\eta_h * v_h\|_{1,\Omega_h}}. \tag{41}$$

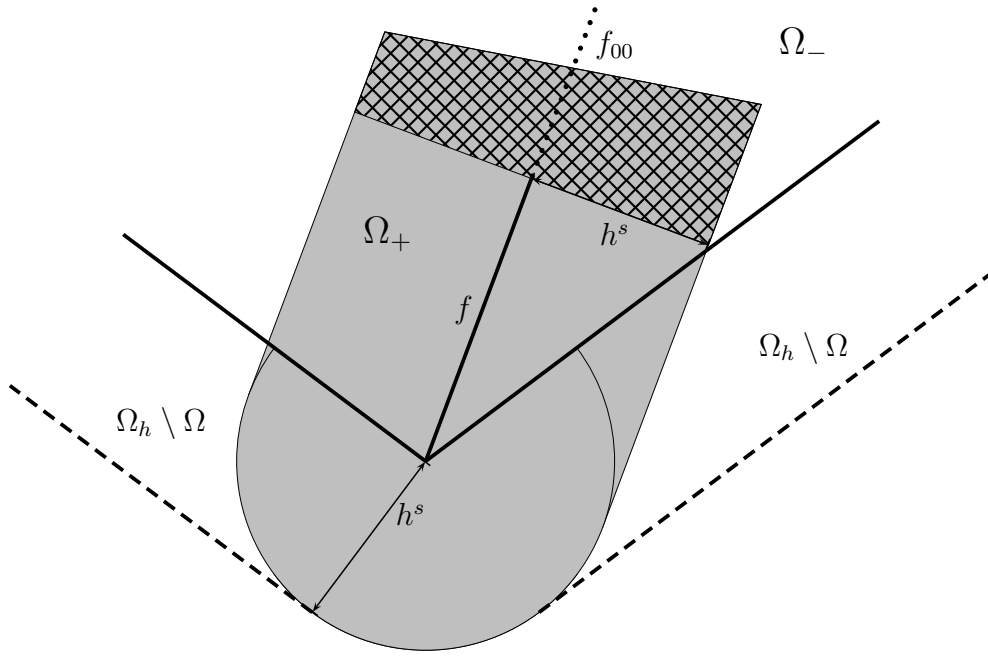


Figure 3: The support of $\eta_h * \llbracket v_h \rrbracket_f$ (shaded) and the set $\{f_{00} \otimes r : r \in [0, h^s]\}$ (with crosshatch) in a 2-dimensional setup.

For the rest, it is sufficient to estimate the second term here. We apply a classical trace inequality in Ω and in $\Omega_h \setminus \Omega$ which imply

$$\begin{aligned}
\sup_{\eta_h * v_h \in \mathbb{P}_{h,k,s}} \frac{\langle \boldsymbol{\nu} \cdot \nabla u, \eta_h * v_h \rangle_{\partial\Omega}}{\|\eta_h * v_h\|_{1,\Omega_h}} &\leq \sup_{\eta_h * v_h \in \mathbb{P}_{h,k,s}} \frac{\|\boldsymbol{\nu} \cdot \nabla u\|_{-\frac{1}{2},\partial\Omega} \|\eta_h * v_h\|_{\frac{1}{2},\partial\Omega}}{\|\eta_h * v_h\|_{1,\Omega_h}} \\
&\lesssim \|u\|_{1,\Omega} \sup_{\eta_h * v_h \in \mathbb{P}_{h,k,s}} \frac{\|\eta_h * v_h\|_{\frac{1}{2},\partial\Omega}}{\|\eta_h * v_h\|_{1,\Omega_h}} \lesssim \|u\|_{1,\Omega} \sup_{\eta_h * v_h \in \mathbb{P}_{h,k,s}} \frac{\|\eta_h * v_h\|_{1,\Omega_h \setminus \Omega}}{\|\eta_h * v_h\|_{1,\Omega_h}} \\
&\lesssim \|u\|_{1,\Omega} \sup_{\eta_h * v_h \in \mathbb{P}_{h,k,s}} \frac{\|\nabla(\eta_h * v_h)\|_{\Omega_h \setminus \Omega}}{\|\nabla(\eta_h * v_h)\|_{\Omega_h}}.
\end{aligned} \tag{42}$$

Observe that the numerator can be rewritten, using to Lemma 1, as

$$\|\nabla(\eta_h * v_h)\|_{\Omega_h \setminus \Omega} = \|\eta_h * \nabla_h v_h + \eta_h * \llbracket v_h \rrbracket\|_{\Omega_h \setminus \Omega} \leq \|\eta_h * \nabla_h v_h\|_{\Omega_h \setminus \Omega} + \|\eta_h * \llbracket v_h \rrbracket\|_{\Omega_h \setminus \Omega}.$$

To analyze the term $\|\eta_h * \llbracket v_h \rrbracket\|_{\Omega_h \setminus \Omega}$ we use the notations corresponding to Lemma 5 and Fig. 2. Furthermore, we define

$$f_{00} = \{\mathbf{x} \in f : d(\mathbf{x}, \partial\Omega) > h^s\}$$

such that

$$\text{supp } \eta_h * \llbracket v_h \rrbracket_f|_{\Omega_h \setminus \Omega} \subset \text{supp } \eta_h * \llbracket v_h \rrbracket_f \setminus \{f_0 \otimes r : r \in [0, h^s]\},$$

see also Fig. 3. The non-degeneracy of \mathcal{T}_h implies that the estimate in the remark after (29) is valid and therefore according to (29) the first term $\frac{1}{B_{h^s,d}^2} \int_{-h^s}^{h^s} I_1(r) dr$ in the second line of (34) provides an upper bound for $\|\eta_h * \llbracket v_h \rrbracket_f\|_{\Omega_h \setminus \Omega}$. Therefore, the estimate in (35) implies

$$\|\eta_h * \llbracket v_h \rrbracket_f\|_{\Omega_h \setminus \Omega}^2 \leq h^{-d} \left[\int_f \llbracket v_h \rrbracket \right]^2 \lesssim h^{-d} h^d h^{s-1} \|\nabla(\eta_h * v_h)\|_{\Omega_+ \cup \Omega_-}^2.$$

Taking their sum for all interelement faces gives then

$$\|\eta_h * \llbracket v_h \rrbracket\|_{\Omega_h \setminus \Omega} \lesssim h^{\frac{s-1}{2}} \|\nabla(\eta_h * v_h)\|_{\Omega}. \quad (43)$$

Note that the condition on non-degeneracy implies that for all subdomains $\Omega_k \subset \tilde{\Omega}_j$ we have

$$\lambda(\Omega_k) \sim h_{\Omega}^d \quad (44)$$

and

$$\lambda(\Omega_{j,h}) \sim h^s h_{\Omega}^{d-1}. \quad (45)$$

Using then (6), (44), (45) and (22) we obtain that

$$\|\eta_h * \nabla_h v_h\|_{\Omega_{j,h}}^2 \leq \lambda(\Omega_{j,h}) \max_{\Omega_j} |\nabla_h v_h|^2 \lesssim \lambda(\Omega_{j,h}) h_{\Omega}^{-d} |\nabla_h v_h|_{\Omega_j}^2 \lesssim h_{\Omega}^{s-1} |\nabla_h v_h|_{\Omega_j}^2 \lesssim h_{\Omega}^{s-1} |\eta_h * \nabla v_h|_{\Omega_j}^2,$$

which can be summed for all subdomain-patches to arrive at

$$\|\eta_h * \nabla_h v_h\|_{\Omega_h \setminus \Omega}^2 \lesssim \sum_{\Omega_j \in \mathcal{T}_h} \|\eta_h * \nabla_h v_h\|_{\Omega_{j,h}}^2 \lesssim h^{s-1} \sum_{\Omega_j \in \mathcal{T}_h} \|\eta_h * \nabla v_h\|_{\Omega_j}^2 \lesssim h^{s-1} \|\eta_h * \nabla v_h\|^2. \quad (46)$$

We can use (43) and (46) to complete the estimation in (42) as

$$\sup_{\eta_h * v_h \in \mathbb{P}_{h,\mathbf{k},s}} \frac{\langle \boldsymbol{\nu} \cdot \nabla u, \eta_h * v_h \rangle_{\partial\Omega}}{\|\eta_h * v_h\|_{1,\Omega_h}} \leq \|u\|_{1,\Omega} \sup_{\eta_h * v_h \in \mathbb{P}_{h,\mathbf{k},s}} \frac{h^{s-\frac{1}{2}} \|\eta_h * \nabla v\|^2}{\|\nabla(\eta_h * v_h)\|_{\Omega_h}} \lesssim \|u\|_{1,\Omega} h^{s-\frac{1}{2}},$$

which together with (41) gives the estimate in the lemma. \square

We easily get now the statement on the convergence of the averaged IP method.

Theorem 3 *The averaged interior penalty approximation is quasi optimal in the following sense:*

$$\|\nabla(u - \eta_h * u_{IP,s})\| \lesssim \inf_{v_h \in \mathbb{P}_{h,\mathbf{k}}} \|u - \eta_h * v_h\|_1 + \mathcal{O}(h^{s-\frac{1}{2}}) + \max_j h_{\Omega_j}^d \|\eta_h * g - g_0\|.$$

Proof: A triangle inequality and the estimates in Theorem 2 and Lemma 7 imply that

$$\begin{aligned} \|\nabla(u - \eta_h * u_{IP,s})\| &\lesssim \|\nabla(u - \eta_h * u_h)\| + \|\nabla(\eta_h * u_{IP,s} - \eta_h * u_h)\| \\ &\lesssim \inf_{v_h \in \mathbb{P}_{h,\mathbf{k}}} \|u - \eta_h * v_h\|_1 + \mathcal{O}(h^{s-\frac{1}{2}}) + \max_j h_{\Omega_j}^d \|\eta_h * g - g_0\|, \end{aligned}$$

as stated in the theorem. \square

Remarks: The above derivation could cover the case of overpenalized IP methods with $s > 1.5$. The increase of the parameter s can lead to ill-conditioned linear problems in the discretizations, such that one should use appropriate preconditioners [3].

Based on the results of the paper, we propose the following introduction of IP methods for the numerical solution of (1).

- Introduce the H^1 -conforming finite element discretization (5).
- Since the $a_{IP,s}$ bilinear form is a lower order approximation of a_{η} and given more explicitly, one should compute $u_{IP,s}$ in the practice.
- Compute the local average $\eta_h * u_{IP,s}$. This converges to the weak solution u of (1) in a quasi optimal way in the H^1 -seminorm.

Appendix

Following the notations in [12] we introduce the smooth function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ with

$$\Phi(\mathbf{x}) = \begin{cases} Ce^{\frac{1}{|\mathbf{x}|^2-1}} & \text{if } |\mathbf{x}| < 1 \\ 0 & \text{if } |\mathbf{x}| \geq 1 \end{cases} \quad \text{and} \quad \int_{B(\mathbf{0},1)} \Phi = 1$$

and define $\Phi_\delta : \mathbb{R}^d \rightarrow \mathbb{R}$ by $\Phi_\delta := \left(\frac{1}{\delta}\right)^d \Phi\left(\frac{\mathbf{x}}{\delta}\right)$. Additionally, for $\mathbf{x} \in \mathbb{R}^d$ we use the notation $\Phi_{\delta,\mathbf{x}}$ for the function given simply by the

$$\Phi_{\delta,\mathbf{x}}(\mathbf{y}) := \Phi_\delta(\mathbf{y}).$$

We use the following proposition; for the proof we refer to [12], pages 713–716.

Proposition 6 *For an arbitrary bounded Lipschitz domain U and parameter $p \in [1, \infty)$ the following statements are valid.*

(i) *For $f \in C(U)$ we have $\lim_{\delta \rightarrow 0} \Phi_\delta * f \rightarrow f$ uniformly.*

(ii) *For any $f \in L_{p,\text{loc}}(U)$ we have $\lim_{\delta \rightarrow 0} \Phi_\delta * f \rightarrow f$ in $L_{p,\text{loc}}(U)$.*

We also need the following statements.

Lemma 8 *If for all $\mathbf{x} \in K_1 \cup K_2$ we have the limit*

$$\lim_{\delta \rightarrow 0} (\eta_h * \llbracket u \rrbracket_f, \Phi_{\delta,\mathbf{x}}) = \tilde{f}(\mathbf{x}) \quad (47)$$

*then $\eta_h * \llbracket u \rrbracket_f$ can be identified with \tilde{f} . Also, for the function $\eta_h * \llbracket u \rrbracket_f : \mathcal{F} \rightarrow \mathbb{R}$ given by*

$$f \ni \mathbf{y} \rightarrow \int_{\mathbb{R}^d} \eta_h(\mathbf{y} - \mathbf{z}) \Phi_{\delta,\mathbf{x}}(\mathbf{z}) \, d\mathbf{z}$$

we have the convergence

$$\lim_{\delta \rightarrow 0} \eta_h * \llbracket u \rrbracket_f \Phi_{\delta,\mathbf{x}} = \eta_h(-\mathbf{x} + \cdot) \quad \text{in } L_1(f). \quad (48)$$

Proof: We first note that

$$\begin{aligned} \langle \eta_h * \llbracket u \rrbracket_f, \Phi_{\delta,\mathbf{x}} \rangle &= \int_{\mathbb{R}^d} \eta_h * \llbracket u \rrbracket_f(\mathbf{x} + \mathbf{y}) \Phi_{\delta,\mathbf{x}}(\mathbf{x} + \mathbf{y}) \, d\mathbf{y} = \int_{\mathbb{R}^d} \eta_h * \llbracket u \rrbracket_f(\mathbf{x} + \mathbf{y}) \Phi_\delta(\mathbf{y}) \, d\mathbf{y} \\ &= \int_{\mathbb{R}^d} \eta_h * \llbracket u \rrbracket_f(\mathbf{x} + \mathbf{y}) \Phi_\delta(-\mathbf{y}) \, d\mathbf{y} = \Phi_\delta * \eta_h * \llbracket u \rrbracket_f(\mathbf{x}). \end{aligned}$$

In this way, according to property (ii) we can rewrite the condition in (47) as

$$\lim_{\delta \rightarrow 0} \Phi_\delta * \eta_h * \llbracket u \rrbracket_f \rightarrow \tilde{f} \quad \text{in } L_{1,\text{loc}}(\mathbb{R}^d). \quad (49)$$

Using the property (i) above, the fact that $\eta_h * \llbracket u \rrbracket_f$ is locally integrable and the limit in (49), we have that for each function $g \in C_0^\infty(\Omega)$ the following equality is valid:

$$\begin{aligned} \langle \eta_h * \llbracket u \rrbracket_f, g \rangle &= (\eta_h * \llbracket u \rrbracket_f, g) = \lim_{\delta \rightarrow 0} (\eta_h * \llbracket u \rrbracket_f, \Phi_\delta * g) = \lim_{\delta \rightarrow 0} (\Phi_\delta * \eta_h * \llbracket u \rrbracket_f, g) \\ &= (\tilde{f}, g), \end{aligned}$$

which proves the first statement of the lemma.

To prove the second statement we rewrite $\eta_h *_f \Phi_{\delta, \mathbf{x}}$ as

$$\eta_h *_f \Phi_{\delta, \mathbf{x}}(\mathbf{y}) = \int_{\mathbb{R}^d} \eta_h(\mathbf{y} - \mathbf{z}) \Phi_\delta(-\mathbf{x} + \mathbf{z}) d\mathbf{z} = \int_{\mathbb{R}^d} \eta_h(\mathbf{y} - \mathbf{x} - \mathbf{z}) \Phi_\delta(\mathbf{z}) d\mathbf{z}.$$

Accordingly, we have the pointwise convergence

$$\eta_h *_f \Phi_{\delta, \mathbf{x}}(\mathbf{y}) \rightarrow \eta_h(\mathbf{y} - \mathbf{x}).$$

On the other hand,

$$\int_{\mathbb{R}^d} \eta_h(\mathbf{y} - \mathbf{x} - \mathbf{z}) \Phi_\delta(\mathbf{z}) d\mathbf{z} \leq \max_{\mathbb{R}^d} |\eta_h|$$

so that the function $\max_{\mathbb{R}^d} |\eta_h| \cdot \mathbb{1} \in L_1(\mathcal{F})$ delivers an upper bound for each function $\eta_h *_f \Phi_{\delta, \mathbf{x}}$. The statement is therefore an obvious consequence of the Lebesgue dominant convergence theorem. \square

Proof of Lemma 5: We compute $\eta_h * \llbracket u \rrbracket_f$ based on the first statement in Lemma 8. For this, we use Lemma 1 and (48), which give

$$\begin{aligned} \lim_{\delta \rightarrow 0} \langle \eta_h * \llbracket u \rrbracket_f, \Phi_{\delta, \mathbf{x}} \rangle &= \lim_{\delta \rightarrow 0} \langle \llbracket u \rrbracket_f, \eta_h * \Phi_{\delta, \mathbf{x}} \rangle = \lim_{\delta \rightarrow 0} \int_f \llbracket u \rrbracket_f(\mathbf{y}) \eta_h *_f \Phi_{\delta, \mathbf{x}}(\mathbf{y}) d\mathbf{y} \\ &= \int_f \llbracket u \rrbracket_f(\mathbf{y}) \lim_{\delta \rightarrow 0} \eta_h *_f \Phi_{\delta, \mathbf{x}}(\mathbf{y}) d\mathbf{y} = \int_f \llbracket u \rrbracket_f(\mathbf{y}) \eta_h(\mathbf{y} - \mathbf{x}) d\mathbf{y} \end{aligned}$$

as stated in the lemma. \square

Proof of Proposition 3 We first prove (17). According to Lemma 1 and the consecutive remark, we have obviously that

$$\int_{f_0} |\llbracket v \rrbracket| \leq |v|_{\text{BV}}, \quad (50)$$

where the BV seminorm is taken on $\overline{K_+ \cup K_-}$. For the next step we use a scaling argument and introduce the function space

$$\bar{\mathbb{P}}_K := \mathbb{P}_{h, \mathbf{k}}|_{K_+ \cup K_-} / \langle \mathbb{1} \rangle,$$

which is the restriction of $\mathbb{P}_{h, \mathbf{k}}$ to $K_+ \cup K_-$ factorized with the constant functions. The BV seminorm on this function space becomes a norm, and accordingly, we use the notation $\|\cdot\|_{\text{BV}}$. We next prove that for all $\epsilon > 0$ there is $h_0 > 0$ such that for all $h < h_0$ and $v \in \bar{\mathbb{P}}_K$ we have

$$\|\eta_h * v - v\|_{\text{BV}} < \epsilon \|v\|_{\text{BV}}. \quad (51)$$

For this we consider a normed basis $\{v_1, v_2, \dots, v_D\}$ of $v \in \bar{\mathbb{P}}_K$ with respect to the BV norm and define the Euclidean norm $\|\cdot\|_E$ generated by this basis such that

$$\left| \sum_{j=1}^D a_j v_j \right|_E^2 = \sum_{j=1}^D a_j^2. \quad (52)$$

This norm should be equivalent with the BV norm, i.e. there is a constant c_0 with

$$\|v\|_E \leq c_0 \|v\|_{\text{BV}} \quad \forall v \in \bar{\mathbb{P}}_{K_0}. \quad (53)$$

Note that this constant should be not the same for all pair of neighboring subdomains, but it is a continuous function of the position of the vertices. In particular, if we fix the edge f_0 of length *one*, then f_0 is fixed for $d = 2$ and for $d = 3$ the remaining vertex should be in a compact set depicted in fig ... if the condition of non-degeneracy holds true. Therefore, the constant c_0 has a finite maximum. Similarly, if we fix now an arbitrary interelement face chosen above the remaining node of K_{0-} and K_{0+} can lie in a compact set. In this way, for each pair of neighboring subdomains with at least one interelement edge of length *one* there is a uniform constant c_0 in (53). Also, since we have a finite basis, and η_h is a Dirac series, there is h_0 such that for all $h < h_0$ we have

$$\|\eta_h * v_j - v_j\|_{\text{BV}} \leq \frac{\epsilon}{c_0 D} \|v_j\|_{\text{BV}} = \frac{\epsilon}{c_0 \sqrt{D}} \quad \forall j \in \{1, 2, \dots, D\}. \quad (54)$$

We obtain also here that (54) is valid for all pair of neighboring subdomains with at least one interelement edge of length *one* with a uniform parameter h_0 . Then using (54), (52) and (53) we have that for any $0 < h < h_0$ and $v = \sum_{j=1}^D a_j v_j \in \mathbb{P}_K$ the following inequality is valid:

$$\begin{aligned} \|\eta_h * v - v\|_{\text{BV}} &= \left\| \sum_{j=1}^D \eta_h * v_j - v_j \right\|_{\text{BV}} \leq \sum_{j=1}^D \|\eta_h * a_j v_j - a_j v_j\|_{\text{BV}} \\ &\leq \frac{\epsilon}{c_0 \sqrt{D}} \sum_{j=1}^D |a_j| \leq \frac{\epsilon}{c_0} \sqrt{\sum_{j=1}^D |a_j|^2} = \frac{\epsilon}{c_0} \|v\|_E \leq \epsilon \|v\|_{\text{BV}}, \end{aligned}$$

which proves the inequality in (51). Consequently, we also have

$$(1 - \epsilon) \|v\|_{\text{BV}} \leq \|v - \eta_h * v\|_{\text{BV}} + \|\eta_h * v\|_{\text{BV}} - \epsilon \|v\|_{\text{BV}} \leq \|\eta_h * v\|_{\text{BV}}.$$

In the last step we relate (50) and (51) and use that $\eta_h * v$ is differentiable to obtain

$$\int_{f_0} \llbracket v \rrbracket \leq \|v\|_{\text{BV}} \leq \frac{1}{1 - \epsilon} \|\eta_h * v\|_{\text{BV}} = \frac{1}{1 - \epsilon} \int_{K_+ \cup K_-} |\nabla(\eta_h * v)|. \quad (55)$$

To prove the statement of the lemma for two arbitrary neighboring subdomains Ω_+ and Ω_- we use (55) and the equalities in (4) which give

$$\begin{aligned} \int_{f_\Omega} \llbracket v \rrbracket &= h_\Omega^{d-1} \int_{f_0} \llbracket v_0 \rrbracket \lesssim h_\Omega^{d-1} \int_{K_+ \cup K_-} |\nabla(\eta_{h_0} * v_0)| \\ &= h_\Omega^{-d} h_\Omega^{d-1} \int_{\Omega_+ \cup \Omega_-} h_\Omega^1 |\nabla(\eta_{h_0 h_\Omega^{\frac{1}{s}}} * v)| = \int_{\Omega_+ \cup \Omega_-} |\nabla(\eta_{h_0 h_\Omega^{\frac{1}{s}}} * v)|. \end{aligned} \quad (56)$$

The inequality remains true if the lower index $h_0 h_\Omega^{\frac{1}{s}}$ is changed to a smaller one since this is equivalent with the choice of a smaller index h_0 . Obviously the condition $h^{1-\frac{1}{s}} < h_0$ implies

$$h_0 h_\Omega^{\frac{1}{s}} > h$$

and using (56) with the previous remark gives that

$$\int_{f_\Omega} \llbracket v \rrbracket \leq \int_{\Omega_+ \cup \Omega_-} |\nabla(\eta_h * v)|$$

as stated in the inequality (17).

To prove (18) we again use the geometric setup discussed at the beginning of Section 4, where for the simplicity, we use the notation $h = h_\Omega$. With this we obtain

$$\begin{aligned}
& \|\eta_h * \nabla u\|_{h \cdot K_+ \cup h \cdot K_-} \left(\int_{h \cdot f_0 \pm h^s} 1 \right)^{\frac{1}{2}} \geq \|\eta_h * \nabla u\|_{h \cdot f_0 \pm h^s} \left(\int_{h \cdot f_0 \pm h^s} 1 \right)^{\frac{1}{2}} \\
& = \|\nabla(\eta_h * u)\|_{h \cdot f_0 \pm h^s} \left(\int_{h \cdot f_0 \pm h^s} 1 \right)^{\frac{1}{2}} \geq \int_{h \cdot f_0 \pm h^s} |\nabla(\eta_h * u)| \\
& \gtrsim \int_{h \cdot f^*} \left| \int_{-h^s}^{h^s} \nabla(\eta_h * u)(x, \mathbf{y}) \, dx \right| \, d\mathbf{y} = \int_{h \cdot f^*} |(\eta_h * u)(h^s, \mathbf{y}) - (\eta_h * u)(-h^s, \mathbf{y})| \, d\mathbf{y} \\
& \geq \int_{h \cdot f^*} |u(h^s, \mathbf{y}) - u(-h^s, \mathbf{y})| - |(\eta_h * u)(h^s, \mathbf{y}) - u(h^s, \mathbf{y})| - |u(-h^s, \mathbf{y}) - (\eta_h * u)(-h^s, \mathbf{y})| \, d\mathbf{y}.
\end{aligned} \tag{57}$$

To continue with the estimate we note that u is differentiable twice in $B((-h^s, \mathbf{y}), h^s)$ and according to (36) we have

$$\begin{aligned}
|(\eta_h * u)(h^s, \mathbf{y}) - u(h^s, \mathbf{y})| \, d\mathbf{y} & \leq \frac{1}{B_{h^s, d}} \cdot \frac{1}{2} \max_{h \cdot K_-} |\nabla^2 u| \int_{B(0, h^s)} |\mathbf{s}|^2 \, d\mathbf{s} \\
& \lesssim \max_{h \cdot K_-} |\nabla^2 u| h^{-sd} h^{s(d+2)} = \max_{h \cdot K_-} |\nabla^2 u| h^{2s}
\end{aligned}$$

Therefore, using (57) we have

$$\|\eta_h * \nabla u\|_{h \cdot K_+ \cup h \cdot K_-} \left(\int_{h \cdot f_0 \pm h^s} 1 \right)^{\frac{1}{2}} \gtrsim \int_{h \cdot f^*} |\llbracket u \rrbracket| - h^{2s} \lambda(h \cdot f^*) \max_{h \cdot K_+ \cup h \cdot K_-} |\nabla^2 u|,$$

which can be rewritten with the aid of (8) and the condition $s \geq \frac{3}{2}$ as

$$\begin{aligned}
\int_{h \cdot f^*} |\llbracket u \rrbracket| & \lesssim h^{2s} h^{d-1} \max_{h \cdot K_+ \cup h \cdot K_-} |\nabla^2 u| + h^{\frac{s}{2} + \frac{d}{2} - \frac{1}{2}} \|\eta_h * \nabla u\|_{h \cdot K_+ \cup h \cdot K_-} \\
& \lesssim h^{2s} h^{d-1} h^{-\frac{d}{2}-1} \|\nabla u\|_{h \cdot K_+ \cup h \cdot K_-} + h^{\frac{s}{2} + \frac{d}{2} - \frac{1}{2}} \|\eta_h * \nabla u\|_{h \cdot K_+ \cup h \cdot K_-} \\
& \lesssim h^{\frac{d}{2}} (h^{\frac{s}{2}-\frac{1}{2}} + h^{2s-2}) \|\eta_h * \nabla u\|_{h \cdot K_+ \cup h \cdot K_-} \lesssim h^{\frac{d}{2}} h^{\frac{s}{2}-\frac{1}{2}} \|\eta_h * \nabla u\|_{h \cdot K_+ \cup h \cdot K_-}
\end{aligned}$$

A summation with respect to the faces gives then the desired inequality. \square

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